

APPROXIMATING ROUGH STOCHASTIC PDES

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ABSTRACT. We study approximations to a class of vector-valued equations of Burgers type driven by a multiplicative space-time white noise. A solution theory for this class of equations has been developed recently in [Hairer, Weber, *Probab. Theory Related Fields*, to appear]. The key idea was to use the theory of *controlled rough paths* to give definitions of weak / mild solutions and to set up a Picard iteration argument.

In this article the limiting behaviour of a rather large class of (spatial) approximations to these equations is studied. These approximations are shown to converge and convergence rates are given, but the limit may depend on the particular choice of approximation. This effect is a spatial analogue to the Itô-Stratonovich correction in the theory of stochastic ordinary differential equations, where it is well known that different approximation schemes may converge to different solutions.

1. INTRODUCTION

The aim of the present paper is to study approximations to vector-valued stochastic Burgers-like equations with multiplicative noise. These equations are of the form

$$\partial_t u = \nu \partial_x^2 u + F(u) + G(u) \partial_x u + \theta(u) \xi, \quad (1.1)$$

where the function $u = u(t, x; \omega) \in \mathbb{R}^n$ is vector-valued. We assume that the functions $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $G, \theta: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ are smooth and the products in the terms $G(u) \partial_x u$ as well as in $\theta(u) \xi$ are to be interpreted as matrix vector multiplication. The noise term ξ denotes an \mathbb{R}^n -valued space-time white noise and the multiplication should be interpreted in the sense of Itô integration against an L^2 -cylindrical Wiener process.

In the case $G = 0$, approximations to (1.1) have been very well studied: we refer to [Gyö98b, Gyö99, DG01] for some of the earlier results in this direction. For non-zero G , there is a clear distinction between the *gradient case*, where $G = \nabla \mathcal{G}$ for some sufficiently regular function $\mathcal{G}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (so that $\partial_t u = G(u) \partial_x u$ would describe a system of conservation laws), and the general case. In the gradient case, existence and uniqueness for (1.1) has been known at least since the nineties [DPDT94, Gyö98a] and convergence results for numerical schemes have, for example, been obtained in [AG06, JB09].

The emphasis of the present article is on the general, non-gradient, case. A satisfactory solution theory for the general case is much more involved than the gradient case and has been given only very recently [Hai12, HW10]. The difficulty

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in treating (1.1) lies in the lack of spatial regularity of its solutions. In fact, it follows from the results in [HW10] that solutions to (1.1) take values in C^α for any $\alpha < \frac{1}{2}$ but not for $\alpha = \frac{1}{2}$. Unfortunately, it turns out that the pairing

$$\mathcal{C}^\alpha \times \mathcal{C}^\alpha \ni (u, v) \mapsto u \partial_x v \quad (1.2)$$

is well-defined if and only if $\alpha > \frac{1}{2}$. Even worse: there exists no “reasonable” Banach space \mathcal{B} containing the solutions to the linearised version of (1.1) and such that (1.2) extends to a continuous bilinear map from $\mathcal{B} \times \mathcal{B}$ into the space of Schwartz distributions, see for example [Lyo91] and [LCL07]. As a consequence, it is not clear at all a priori how to interpret the term $G(u) \partial_x u$ in (1.1) and the classical approach to the construction of mild solutions fails.

In all of the above mentioned references on the gradient case, this issue is resolved by exploiting the conservation law structure of the nonlinearity. This means that the chain rule is *postulated* and the nonlinearity is rewritten as

$$G(u(t, x)) \partial_x u(t, x) = \partial_x \mathcal{G}(u(t, x)) , \quad (1.3)$$

which makes sense as a distribution as soon as u is continuous. The approximation schemes studied e.g. in [AG06, JB09] respect this conservation law structure by considering natural approximations of $\partial_x \mathcal{G}$. For example, it is not difficult to show that if u_ε solves

$$\partial_t u_\varepsilon = \nu \partial_x^2 u_\varepsilon + F(u_\varepsilon) + \frac{1}{\varepsilon} \left(\mathcal{G}(u_\varepsilon(t, x + \varepsilon)) - \mathcal{G}(u_\varepsilon(t, x)) \right) + \theta(u_\varepsilon) \xi , \quad (1.4)$$

then u_ε converges to u for $\varepsilon \downarrow 0$. Similarly, full finite difference / element approximations also converge.

In the *non-gradient* case, i.e. when such a function \mathcal{G} does not exist, this approach does not work. The key idea developed in [Hai12, HW10] to overcome this difficulty is the following: in order to define the product $G(u(t, x)) \partial_x u(t, x)$ as a distribution, it has to be tested against a smooth test function φ . This expression takes the form

$$\int_{-\pi}^{\pi} \varphi(x) G(u(t, x)) \partial_x u(t, x) dx = \int_{-\pi}^{\pi} \varphi(x) G(u(t, x)) d_x u(t, x). \quad (1.5)$$

The fact that we expect u to behave like a Brownian motion as a function of the space variable x suggests that one should interpret this expression as a kind of stochastic integral. In particular, a stochastic integration theory is needed to capture stochastic cancellations. It turns out that the theory of *controlled rough paths* [Lyo98, LQ02, LCL07, Gub04, FV10, GT10] provides a suitable way to deal with spatial stochastic integrals like (1.5). Using this idea, a concept of solutions is given in [Hai12, HW10]. These solutions exist and are unique up to a choice of *iterated integral* which corresponds to the choice of the integral of u against itself. This is a situation analogous to the choice between Itô and Stratonovich integral that is familiar from the classical theory of SDEs.

Even in the gradient case, effects of this non-uniqueness can be observed. A posteriori, this is not surprising: after some reflection, it clearly appears that postulating the chain rule (1.3) is a rather bold step to take! Indeed, we have just seen that the expression (1.5) is akin to a stochastic integral, and we know very well that the usual chain rule only holds if such an integral is interpreted in the Stratonovich sense, while it fails if it is interpreted in the Itô sense. In [HM10] approximations to (1.1) are studied in the special case $G = \nabla \mathcal{G}$ when the noise is additive, i.e. if

$\theta(u) = 1$. For a whole class of different natural approximation schemes, convergence to a stochastic process \bar{u} is shown. The main difference with previous works is that in [HM10], natural discretisations of $G(u) \partial_x u$ instead of natural discretisations of $\partial_x \mathcal{G}(u)$ are considered. A typical example of the type of discretisation for the nonlinearity considered there is

$$G(u(t, x)) \frac{1}{\varepsilon} (u(t, x + \varepsilon) - u(t, x)). \quad (1.6)$$

In general, the limiting process $\bar{u} = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$ turns out *not* to be a solution of (1.1) in the classical sense. Instead, it solves a similar equation with an additional reaction term. This extra term depends on the specific choice of approximation and it can be calculated explicitly. As noted in [HM10], this additional term is exactly the correction that appears when changing to a different stochastic integral.

In the present work, these approximation results are extended to the *non-gradient* case with *multiplicative* noise. We study a wide class of approximations (essentially the same as in [HM10] but with slightly different technical assumptions) and extend the convergence result to the general case. Unsurprisingly, the techniques we use are quite different from [HM10], since the notion of solution for the limiting object is completely different. We make full use of the machinery developed in [Hai12, HW10] and we develop a method to include approximations to rough integrals. In particular, we do obtain an explicit rate of convergence of the order $\varepsilon^{\frac{1}{6}-\kappa}$ for κ arbitrarily small.

There are several motivations for this work: Equation (1.1) appears, for example, in the path sampling algorithm introduced in [HSV07] (see also [Hai12]). So far, the fact that the limit depends on the specific choice of approximation scheme had been shown only in the gradient case with additive noise. In this work we complete the picture by showing that the same effect can be observed in the general case and obtaining an expression for the correction term that arises.

Another main motivation is to illustrate how the rough path machinery can be used to obtain concrete approximation results, including convergence rates. This is particularly interesting, as similar techniques were recently used in [Hai11b] to give a solution theory for the KPZ equation [KPZ86]

$$\partial_t h = \partial_x^2 h + \lambda (\partial_x h)^2 - \infty + \xi,$$

where ξ denotes space-time white noise and “ ∞ ” denotes an “infinite constant” that needs to be subtracted in order to make sense of the diverging term $(\partial_x h)^2$. This equation is a popular model for surface growth (see e.g. [Cor11] and the references therein). It is conjectured that a large class of microscopic surface growth models (e.g. the lattice KPZ equation [SS09] and variations on the weakly asymmetric simple exclusion process [GJ10, Ass11]), converge to h in suitable scaling limits, but so far this has only been shown for the weakly asymmetric simple exclusion process [BG97].

The present article provides a case study illustrating how one can obtain approximation results for a class of equations exhibiting similar features to those of the KPZ equation (see [Hai11b, Section 4]). In this sense, the present work is really a “proof of concept” that lays the foundations for further analytical investigations into the universality of the KPZ equation. Notice that although the KPZ equation has additive noise, the construction in [Hai11b] yields an equation that is very close to the case of multiplicative noise treated here.

1.1. Framework and main result. For $\varepsilon > 0$ we consider a class of approximating stochastic PDEs given by

$$\begin{aligned} du_\varepsilon &= \left(\nu \Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + G(u_\varepsilon) D_\varepsilon u_\varepsilon \right) dt + \theta(u) H_\varepsilon dW \\ u_\varepsilon(0) &= u_\varepsilon^0. \end{aligned} \quad (1.7)$$

Here, as usual, we have replaced the formal ξ with the stochastic differential of a cylindrical Brownian motion W on L^2 . The integral against dW should furthermore be interpreted in the Itô sense. For simplicity, we assume that x takes values in $[-\pi, \pi]$ and we endow (1.7) with periodic boundary conditions. We do not expect our results to significantly depend on this choice. Throughout the paper we will assume that $F \in \mathcal{C}^1$, $G \in \mathcal{C}^3$, and $\theta \in \mathcal{C}^2$.

The operators Δ_ε , D_ε , and H_ε appearing in (1.7) are Fourier multipliers providing approximations to ∂_x^2 , ∂_x and the identity respectively. In terms of their action in Fourier space, they are given by

$$\widehat{\Delta_\varepsilon u}(k) = -k^2 f(\varepsilon k) \widehat{u}(k), \quad (1.8a)$$

$$\widehat{D_\varepsilon u}(k) = ik g(\varepsilon k) \widehat{u}(k), \quad (1.8b)$$

$$\widehat{H_\varepsilon W}(k) = h(\varepsilon k) \widehat{W}(k). \quad (1.8c)$$

Throughout the paper we will make some standing assumptions on the cut-off functions f , g and h .

Assumption 1.1. *The function $f : \mathbb{R} \rightarrow (0, +\infty]$ is even, satisfies $f(0) = 1$, and is continuously differentiable on an interval $[-\delta, \delta]$ around 0. Furthermore, there exists $c_f \in (0, 1)$ such that $f(k) \geq 2c_f$ for all $k > 0$.*

Besides this weak regularity assumption of f near the origin, we also need a global bound on its oscillations. In order to state this bound, we introduce the family of functions

$$b_t(k) = \exp \left(-k^2 (f(k) - c_f) t \right).$$

With this notation at hand, we assume that

Assumption 1.2. *The functions b_t are uniformly bounded in the bounded variation norm:*

$$\sup_{t>0} |b_t|_{\text{BV}} < \infty.$$

We make the following assumption on the approximation of the spatial derivative.

Assumption 1.3. *There exists a signed Borel measure μ such that*

$$\int_{\mathbb{R}} e^{ikx} \mu(dx) = ik g(k),$$

and such that

$$\mu(\mathbb{R}) = 0, \quad |\mu|(\mathbb{R}) < \infty, \quad \int_{\mathbb{R}} x \mu(dx) = 1, \quad \int_{\mathbb{R}} |x|^2 |\mu|(dx) < \infty. \quad (1.9)$$

In particular, we have $(D_\varepsilon u)(x) := \frac{1}{\varepsilon} \int_{\mathbb{R}} u(x + \varepsilon y) \mu(dy)$, where we identify $u : [-\pi, \pi] \rightarrow \mathbb{R}$ with its periodic extension on all of \mathbb{R} .

Note that the case $D_\varepsilon u(x) = \frac{1}{\varepsilon}(u(x+\varepsilon) - u(x))$ mentioned in (1.4) and (1.6) is included as special case $\mu = \delta_1 - \delta_0$. Finally, we make the following assumption on the approximation of the noise.

Assumption 1.4. *The function h is even, bounded and so that h^2/f is of bounded variation. Furthermore, h is twice differentiable at the origin with $h(0) = 1$ and $h'(0) = 0$.*

Note that the assumptions on g and h are identical to those imposed in [HM10]. Regarding the function f , Assumption 1.1 is actually weaker than the corresponding assumption in [HM10]. However, we require the additional Assumption 1.2. This assumption is not too restrictive and in particular all the examples discussed in [HM10] satisfy it. See Remark 1.9 below for the main reason why this additional assumption is required. Note that the assumptions on f do not imply that the approximated heat semigroup $S_\varepsilon(t) := e^{t\Delta_\varepsilon}$ is continuous at 0 in the space of continuous functions. This is natural in the context of numerical approximations, since these would always involve the projection onto a finite-dimensional subspace. See Subsections 2.2 and 2.3 below for a more detailed discussion of this point.

Let \bar{u} be the solution of the equation

$$\begin{aligned} d\bar{u} &= \left(\nu \partial_x^2 \bar{u} + \bar{F}(\bar{u}) + G(\bar{u}) \partial_x \bar{u} \right) dt + \theta(\bar{u}) dW, \\ \bar{u}(0) &= u_0. \end{aligned} \quad (1.10)$$

In this equation, the vector valued function F is given by

$$\bar{F}^i := (F^i - \Lambda \theta_k^j \partial_j G_l^i \theta_k^l), \quad (1.11)$$

where we follow the convention to sum over repeated indices. The correction constant Λ can be calculated explicitly as

$$\Lambda \stackrel{\text{def}}{=} \frac{1}{2\pi\nu} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{(1 - \cos(yt))h^2(t)}{t^2 f(t)} \mu(dy) dt. \quad (1.12)$$

Note that a straightforward calculation shows that Λ is indeed well-defined, as a consequence of the fact that $h^2 \lesssim f$ by assumption and that $|\mu|$ has a finite second moments. The constant Λ is identical to the constant appearing in [HM10]. There, it has been calculated for several natural approximation schemes including the case where only the nonlinearity is discretised, as well as a finite difference and a Galerkin discretisation.

Note that in the non-gradient case $G \neq \nabla \mathcal{G}$, (1.10) has to be interpreted as in [HW10]. Actually, there a slightly different equation is considered - the equation studied in [HW10] does not include the reaction term \bar{F} and more importantly, global boundedness of G, θ as well as its derivatives up to order three is assumed to guarantee global existence. Treating the additional reaction term \bar{F} is a straightforward modification that does not pose any problem for this approach. In the present paper, we also drop the assumption on the boundedness of F, G and θ , so we allow for explosion in finite time. We will simply deal with this by working up to a suitable stopping time. More precisely, for any $K > 0$ we define the stopping times

$$\tau_K^* := \inf \{t: |\bar{u}(t)|_C \geq K\},$$

where $|\cdot|_C$ denotes the supremum norm. The explosion time of \bar{u} is then defined to be $\tau^* = \lim_{K \rightarrow \infty} \tau_K^*$.

The main result of this article is the following theorem.

Theorem 1.5. *Let $\alpha_\star = \frac{1}{2} - \kappa$ for some $\kappa > 0$. Then, for every κ small enough, there exists a $\gamma > 0$ with $\lim_{\kappa \rightarrow 0} \gamma(\kappa) = \frac{1}{6}$ such that the following is true.*

Let $|u^0|_{C^{\alpha_\star}} < \infty$ and $\sup_{\varepsilon \leq 1} |u_\varepsilon^0|_{C^{\alpha_\star}} < \infty$ and denote by u_ε and \bar{u} the solutions to (1.7) and (1.10). If the initial data u_ε^0 and u^0 satisfy additionally

$$|u_\varepsilon^0 - u^0|_{C^{1/3}} \lesssim \varepsilon^\gamma,$$

then there exists a sequence of stopping times τ_ε satisfying $\lim_{\varepsilon \rightarrow 0} \tau_\varepsilon = \tau^$ in probability, and such that for any $\tilde{\gamma} < \gamma$*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left(\sup_{0 \leq t \leq \tau_\varepsilon} |u_\varepsilon(t) - \bar{u}(t)|_C > \varepsilon^{\tilde{\gamma}} \right) = 0.$$

Remark 1.6. As pointed out below in Section 2 and in Appendix A the construction of the integral $\int \varphi G(u) du$ involves in principle the choice of *iterated integrals* of a certain Gaussian process, but there turns out to exist a canonical choice \mathbf{X} . The solution theory developed in [HW10] still works if we replace \mathbf{X} by

$$\tilde{\mathbf{X}}(s; x, y) \stackrel{\text{def}}{=} \mathbf{X}(s; x, y) - \Lambda(y - x) \text{Id},$$

but it yields a *different* solution. In [HW10] it was shown that this solution then coincides with \bar{u} . One can interpret this as stating that the approximations u_ε converge to solutions of the *correct* equation (1.1), but where a *different* stochastic integral is used to interpret the nonlinearity involving G .

Remark 1.7. In the additive noise case our rate of convergence is not optimal. Actually, at least in the case where the noise is additive and one only discretises the derivative, our argument in Sections 4 and 5 would give a better rate. We believe that in that case a slight improvement of our calculations would yield a rate of almost $\varepsilon^{1/2}$. We suspect this to be the true rate of convergence in that case.

In the multiplicative case we do not expect the convergence to be very quick and our rate could be close to optimal. Actually, in [HV11] approximations to (1.1) were studied numerically. In the case of additive noise the convergence which is the content of Theorem 1.5 could be observed, but not in the case of multiplicative noise. It might however be possible to improve the rate of convergence by considering weak (in the probabilistic sense) convergence, as was observed in [TT90] and recently exploited in the approximation to (1.1) when $G = 0$ [Deb11].

Note also that the rate $\frac{1}{6}$ obtained here seems unrelated to the “order barrier” mentioned in [DG01].

Remark 1.8. The condition that the initial conditions are bounded in C^{α_\star} and converge in a larger space $C^{\frac{1}{3}}$ may seem slightly bulky. We choose to state the result in this way to obtain the optimal rate of convergence. Note that if u^0 has the regularity of Brownian motion and u_ε^0 is a piecewise linearisation, then these conditions are satisfied. We also refer to remark 2.1 for a more detailed discussion about the initial condition.

Remark 1.9. A crucial technical difference between the present article and [HM10] comes from the fact that for most of the argument we work in Hölder spaces instead of Sobolev spaces. This is necessary to apply the theory of controlled rough paths. Some arguments become easier in Hölder spaces because Gaussian random fields tend to have the same degree of Hölder regularity as Sobolev regularity. The

sample paths of Brownian motion, for example, take values in every Sobolev space H^s for $s < \frac{1}{2}$ but in no H^s for $s \geq \frac{1}{2}$. It also takes values in C^α for the same values of α which is a much stronger statement. (Sobolev embedding would not even yield continuous sample paths!) Using this additional information, we can skip the messy *high frequency cut-off* needed in the proof in [HM10]. The price to pay is that it is more difficult to get bounds on the approximated heat semigroup. As the approximations are given in Fourier coordinates, bounds in L^2 -based Sobolev spaces are trivial to obtain, but the derivation in Hölder spaces requires some work. For example, we need the additional Assumption 1.2 to ensure that the approximations of the heat semigroup are well-behaved not only in Sobolev but also in Hölder spaces.

Remark 1.10. It is always possible to reduce ourselves to the case $\nu = 1$ by performing a simple time change. For the sake of conciseness, we therefore make this choice throughout the remainder of this article.

1.2. Structure of the paper. We start Section 2 with a short reminder of the solution theory from [HW10]. Then we introduce the main quantities needed for the proof of Theorem 1.5 and state the bounds on these. Finally, at the end of this section we give the proof of our main result. In the remaining sections we give the proofs for the bounds stated in Section 2. In Section 3 we provide a priori bounds on the main quantities involved. In Section 4 the convergence of the *extra term* is proved. In Section 5 the convergence of the term involving the spatial rough integrals is shown. The Sections 3 – 5 form the core of our argument. In Section 6 we prove some auxiliary regularity results. Finally, in Appendix A we recall some basic notions of rough path theory used in this work and in Appendix B we give a higher-dimensional extension of the classical Garsia-Rodemich-Rumsey Lemma.

1.3. Norms and notation. Throughout the paper we will use a whole zoo of different Hölder type norms and for later reference we provide a list here. For a normed vector space V we denote by $\mathcal{C}(V)$ the space of continuous functions from $[-\pi, \pi]$ to V and by $\mathcal{B}(V)$ the space of continuous functions from $[-\pi, \pi]^2$ to V vanishing on the diagonal (i.e. for $R \in \mathcal{B}(V)$ we have $R(x, x) = 0$ for all $x \in [-\pi, \pi]$). We will often omit the reference to the space V when it is clear from the context and simply write \mathcal{C} and \mathcal{B} instead.

For a given parameter $\alpha \in (0, 1)$ we define Hölder-type semi-norms:

$$|X|_\alpha = \sup_{x \neq y} \frac{|X(x) - X(y)|}{|x - y|^\alpha} \quad \text{and} \quad |R|_\alpha = \sup_{x \neq y} \frac{|R(x, y)|}{|x - y|^\alpha}, \quad (1.13)$$

and denote by \mathcal{C}^α resp. \mathcal{B}^α the set of functions for which these semi-norms are finite. The space \mathcal{C}^α endowed with $|\cdot|_{\mathcal{C}^\alpha} = |\cdot|_{\mathcal{C}} + |\cdot|_\alpha$ is a Banach space. Here $|\cdot|_{\mathcal{C}}$ denotes the supremum norm. The space $\mathcal{B}^\alpha(V)$ is a Banach space endowed with $|\cdot|_\alpha$ alone.

For a function $u : [0, T] \times [-\pi, \pi] \rightarrow \mathbb{R}^n$ or $u : [0, T] \times [-\pi, \pi] \rightarrow \mathbb{R}^{n \times n}$ and for any $\alpha_1, \alpha_2 \in (0, 1)$ and $t_1 < t_2 \leq T$ we denote by

$$\|u\|_{\mathcal{C}_{[t_1, t_2]}^{\alpha_1, \alpha_2}} := \sup_{\substack{s_1, s_2 \in [t_1, t_2] \\ x, y \in [-\pi, \pi]}} \frac{|u(s_1, x) - u(s_2, y)|}{|s_1 - s_2|^{\alpha_1} + |x - y|^{\alpha_2}} + \sup_{\substack{s \in [t_1, t_2] \\ x \in [-\pi, \pi]}} |u(s, x)| \quad (1.14)$$

the inhomogeneous α_1, α_2 -Hölder norm of u . In most cases we will have $t_1 = 0$ and then we simply write

$$\|u\|_{\mathcal{C}_t^{\alpha_1, \alpha_2}} := \|u\|_{\mathcal{C}_{[0, t]}^{\alpha_1, \alpha_2}}. \quad (1.15)$$

If we are only interested in the spatial regularity, we write for $\gamma \in (0, 1)$

$$\|u\|_{\mathcal{C}_{[t_1, t_2]}^\gamma} := \sup_{\substack{s \in [t_1, t_2] \\ x, y \in [-\pi, \pi]}} \frac{|u(s, x) - u(s, y)|}{|x - y|^\gamma} + \sup_{\substack{s \in [t_1, t_2] \\ x \in [-\pi, \pi]}} |u(s, x)|, \quad (1.16)$$

and if $t_1 = 0$ we use $\|u\|_{\mathcal{C}_t^\gamma} := \|u\|_{\mathcal{C}_{[0, t]}^\gamma}$. We simply write

$$\|u\|_{\mathcal{C}_{[t_1, t_2]}} := \sup_{s \in [t_1, t_2]} \sup_{x \in [-\pi, \pi]} |u(s, x)| \quad (1.17)$$

and $\mathcal{C}_t := \mathcal{C}_{[0, t]}$ for the supremum norm. We will also need a similar norm, for functions that depend on two space variables. For $R : [0, T] \times [-\pi, \pi]^2 \rightarrow \mathbb{R}^n$ or $R : [0, T] \times [-\pi, \pi]^2 \rightarrow \mathbb{R}^{n \times n}$ we write

$$\|R\|_{\mathcal{B}_{[t_1, t_2]}^\gamma} := \sup_{s \in [t_1, t_2]} \sup_{x \in [-\pi, \pi]} \frac{|R(s; x, y)|}{|x - y|^\gamma}. \quad (1.18)$$

Finally, we will sometimes have to allow for blowup of a function near time $t_1 \geq 0$. This can be captured by

$$\|R\|_{\mathcal{B}_{[t_1, t_2], \beta}^\gamma} := \sup_{s \in (t_1, t_2]} (s - t_1)^\beta \sup_{x, y \in [-\pi, \pi]} \frac{|R(s; x, y)|}{|x - y|^\gamma}, \quad (1.19)$$

for some $\beta \in [0, 1]$. As above, if $t_1 = 0$ we write

$$\|R\|_{\mathcal{B}_{t, \beta}^\gamma} := \|R\|_{\mathcal{B}_{[0, t], \beta}^\gamma}. \quad (1.20)$$

We will write $\mathcal{C}_{[t_1, t_2]}^{\alpha_1, \alpha_2}$, $\mathcal{C}_{[t_1, t_2]}^\gamma$, $\mathcal{C}_{[t_1, t_2]}$, $\mathcal{B}_{[t_1, t_2]}^\gamma$ and $\mathcal{B}_{[t_1, t_2], \beta}^\gamma$ for the spaces of continuous functions for which these norms are finite.

We will avoid the use of indices as much as possible and only use them if expressions would get ambiguous otherwise. When we do use indices, we always use the convention of summation over repeated indices. We will write $A^+ = \frac{1}{2}(A + A^*)$ and $A^- = \frac{1}{2}(A - A^*)$ for the symmetric and anti-symmetric part of a matrix A . The Hilbert-Schmidt norm of a matrix A will simply be denoted by $|A|$.

Finally, we will use the notation $x \lesssim y$ to indicate that there exists a constant C that does not depend on the relevant quantities so that $x \leq C y$. Similarly, $x \approx y$ means that $C^{-1}x \leq y \leq Cx$.

2. OUTLINE AND PROOF OF THE MAIN RESULT

We start this section by presenting an outline of the construction of solutions to (1.1) in Subsection 2.1. In Subsection 2.2 we discuss how the quantities involved behave under approximations. The proofs of the bounds announced in this subsection form the core of this article and will be presented in the subsequent sections. Finally, in Subsection 2.3 these bounds will be summarised to give a proof of Theorem 1.5.

2.1. Construction of solutions to rough Burgers-like equations. In this section we give an outline of the construction of local solutions to (1.1). The construction given here differs slightly from the construction presented in [HW10]. This will make the proof of the main result in Subsection 2.3 more transparent. We comment on the differences below in Remark 2.2. We refer the reader to Appendix A for the necessary notions of rough path theory. For the moment, we assume that α is an arbitrary exponent in $(\frac{1}{3}, \frac{1}{2})$. We will fix it below in Subsection 2.3.

Let us start by fixing some notation. Throughout the paper we will write

$$S(t) = e^{t\Delta}$$

for the semigroup generated by Δ . Recall that the operator $S(t)$ acts on functions as convolution (on the torus) with the heat kernel

$$p_t(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} e^{-tk^2} e^{ikx}. \quad (2.1)$$

For adapted $L^2[-\pi, \pi]$ valued processes θ and F we will frequently write

$$\Psi^\theta(t) := \int_0^t S(t-s) \theta(s) dW(s) \quad \text{and} \quad \Phi^F(t) := \int_0^t S(t-s) F(s) ds.$$

Then with this notation the *mild* formulation of (1.1) reads

$$u(t) = S(t) u^0 + \Psi^{\theta(u)}(t) + \Phi^{F(u)}(t) + \int_0^t S(t-s) G(u(s)) \partial_x u(s) ds. \quad (2.2)$$

The terms $S(t)u^0$ and the reaction term $\Phi^{F(u)}$ do not cause any major difficulty. Therefore, for the moment we will concentrate on the two terms

$$\Psi^{\theta(u)} = \int_0^t S(t-s) \theta(u(s)) dW(s) \quad \text{and} \quad \int_0^t S(t-s) G(u(s)) \partial_x u(s) ds.$$

As pointed out in the introduction, we will use the theory of controlled rough paths to interpret the term involving G . We will write

$$\begin{aligned} \Xi^u(t) &:= \int_0^t S(t-s) G(u(s)) \partial_x u(s) ds \\ &:= \int_0^t \left[\int_{-\pi}^{\pi} p_{t-s}(\cdot - y) G(u(s, y)) dy u(s, y) \right] ds. \end{aligned} \quad (2.3)$$

In order to define the spatial integral on the right-hand side of (2.3) as a rough integral, for every $s \in (0, t)$ we must specify a reference path $X(s)$. These reference paths must meet the following requirements:

- For every s it must be possible to construct the iterated integrals

$$\mathbf{X}(s; x, y) = \int_x^y (X(s, z) - X(s, y)) \otimes d_z X(s, z). \quad (2.4)$$

- For every s the random function $x \mapsto u(s, x)$ must be controlled by $X(s)$, in the sense that we need to be able find a derivative process $u'(s, x)$ such that

$$u(s, y) - u(s, x) = u'(s, x)(X(s, y) - X(s, x)) + R_u(s; x, y), \quad (2.5)$$

where the remainder R_u is more regular than u .

Such reference paths are provided by the (Gaussian) solution to the linear stochastic heat equation

$$\partial_t X = \partial_x^2 X + \xi.$$

Actually, the construction of \mathbf{X} is rather straightforward. The point is that X is a Gaussian process with explicitly known covariance structure so that known existence results (see [FV10]) apply. The process \mathbf{X} is constructed by evaluating (2.4) for a sequence of approximations to X and checking that the sequence of approximate iterated integrals converges in the right sense. The crucial ingredient for this calculation is provided by Nelson's estimate that yields the equivalence of all moments in a given Wiener chaos.

When checking (2.5) it is sufficient to look at the term $\Psi^{\theta(u)}$. Actually, the terms $S(t)u^0$ and $\Phi^{F(u)}$ will be \mathcal{C}^1 in space. So they can be included in the remainder R^u and we need not worry about them. The same is true for the term Ξ^u discussed in (2.3) as can be established with the *scaling lemma* A.5.

For $\Psi^{\theta(u)}$ we can write

$$\Psi^{\theta(u)}(t, y) - \Psi^{\theta(u)}(t, x) = \theta(t, x)(X(t, y) - X(t, x)) + R^{\theta(u)}(t; x, y).$$

It is shown in [HW10, Proposition 4.8] that the term $R^{\theta(u)}$ does indeed have the necessary 2α regularity near the diagonal as soon as

$$\mathbb{E} \left(\sup_{\substack{s, t \in [0, \tau] \\ x, y \in [-\pi, \pi]}} \frac{|u(s, x) - u(t, y)|}{|t - s|^{\alpha/2} + |x - y|^\alpha} + \sup_{\substack{t \in [0, \tau] \\ x \in [-\pi, \pi]}} |u(t, x)| \right)^p \quad (2.6)$$

is finite for a suitable stopping time τ and large enough p . This is precisely the regularity we expect for u . With these observations at hand we are ready to set up a fixed point argument to solve (2.2).

We set $\beta = \alpha + \kappa/3$, where as above $\kappa > 0$ is assumed to be sufficiently small. Then for some $p \geq 2$ we denote by \mathcal{A}_p the space of triples

$$(u, u', R_u) \in L^p(\mathcal{C}_T^{\alpha/2, \alpha}) \times L^p(\mathcal{C}_T^\alpha) \times L^p(\mathcal{B}_{T, \beta/2}^{2\alpha})$$

that satisfy the following conditions:

- The processes $t \mapsto u$, $t \mapsto u'$, $t \mapsto R_u$ are adapted.
- Almost surely, for every $t \in [0, T]$ the triple $u(t, \cdot)$, $u'(t, \cdot)$, $R_u(t, \cdot)$ is controlled by $X(t, \cdot)$. To be more precise, we assume that (2.5) holds almost surely for all s, x, y .

Here the L^p refers to p -th stochastic moments.

It is easy to check that \mathcal{A}_p is a closed linear subspace of $L^p(\mathcal{C}_T^{\alpha/2, \alpha}) \times L^p(\mathcal{C}_T^\alpha) \times L^p(\mathcal{B}_{T, \beta/2}^{2\alpha})$. Then for such a triple (u, u', R_u) and for any stopping time $\tau \leq T$ it makes sense to define

$$\mathcal{M} : (u, u', R_u) \mapsto (\tilde{u}, \tilde{u}', R_{\tilde{u}}),$$

where

$$\begin{aligned} \tilde{u}(t) &:= \left(S u^0 + \Psi^{\theta(u)} + \Phi^{F(u)} + \Xi^u \right)(t \wedge \tau), \\ \tilde{u}'(t) &:= \theta(u(t \wedge \tau)), \\ R_{\tilde{u}}(t) &:= \delta \left(S u^0 + \Phi^{F(u)} + \Xi^u \right)(t \wedge \tau) + \left(\delta \Psi^{\theta(u)} - \theta(u) \delta X \right)(t \wedge \tau). \end{aligned} \quad (2.7)$$

Here the difference operator δ is defined as

$$\delta u(s; x, y) := u(s, y) - u(s, x). \quad (2.8)$$

Using the bounds mentioned above, we can show that, for $u^0 \in \mathcal{C}^\alpha$ and under suitable assumptions on τ, κ and p , the operator \mathcal{M} is a contraction from a ball in \mathcal{A}_p into itself.

As usual one can obtain solutions on a longer time interval by iterating this procedure. For fixed choice of the iterated integral process \mathbf{X} these solutions are unique.

Remark 2.1. The reason for allowing the remainder $R^{\bar{v}}$ to blow up like $t^{-\beta/2}$ near zero lies in the initial condition $u^0 \in \mathcal{C}^\alpha$. Actually, the regularising property of the heat semigroup implies that for $\beta \geq \alpha$ we have

$$\sup_t t^{\frac{\beta}{2}} |S(t)u^0|_{\mathcal{C}^{2\alpha}} \lesssim |u^0|_{\mathcal{C}^\alpha}.$$

We need this bound to control the contribution of the initial condition to the remainder term.

This issue would be avoided completely if we could assume that $u^0 \in \mathcal{C}^{2\alpha}$. The problem is that even under this stronger assumption on the initial condition, after positive time the solutions $u(t)$ would only attain values in \mathcal{C}^α for any $\alpha < \frac{1}{2}$. This would make it impossible to iterate this construction to get solutions on a longer time interval.

Another way of avoiding the explosion of the remainder would be to assume that u^0 is already controlled by a Gaussian process. We could, for example, replace the reference rough paths X by the stationary solution \bar{X} to the stochastic heat equation with an additional potential. Then it would make sense to assume that u_0 is controlled by $\bar{X}(0)$.

Remark 2.2. The construction in [HW10] is slightly different as it is split up into an *inner* fixed point argument to deal with the term involving G and an *outer* fixed point argument to conclude. This corresponds to a semi-implicit Picard iteration

$$\begin{aligned} u_{n+1}(t) &= S(t)u^0 + \int_0^t S(t-s)\theta(u_n(s))dW(s) \\ &\quad + \int_0^t S(t-s)F(u_n(s))ds + \int_0^t S(t-s)G(u_{n+1}(s))\partial_x u(s)ds. \end{aligned}$$

The advantage of this approach is that it separates more clearly the deterministic part from the probabilistic part of the construction. The price to pay is that some stopping arguments get more involved. In terms of the bounds needed both constructions are essentially equivalent.

2.2. Outline: Behaviour of the main quantities under approximation. In order to prove Theorem 1.5 we will go through the construction we just described and see how the terms behave under approximation.

For $\varepsilon > 0$ we denote by $S_\varepsilon(t) = e^{t\Delta_\varepsilon}$ the semigroup generated by the approximated Laplacian defined in (1.8a). Similarly to $S(t)$, it is given by convolution (on the torus) with a heat kernel

$$p_t^\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} e^{-tk^2 f(\varepsilon k)} e^{ikx}. \quad (2.9)$$

As above, for any adapted $L^2[-\pi, \pi]$ valued processes θ and F we will write

$$\Psi_\varepsilon^\theta(t) = \int_0^t S_\varepsilon(t-s) \theta(s) H_\varepsilon dW(s) \quad \text{and} \quad \Phi_\varepsilon^F(t) = \int_0^t S_\varepsilon(t-s) F(s) ds.$$

Using this notation the mild version of the approximation (1.7) takes the form

$$\begin{aligned} u_\varepsilon(t) &= S_\varepsilon(t) u_\varepsilon^0 + \Psi_\varepsilon^{\theta(u_\varepsilon)}(t) + \Phi_\varepsilon^{F(u_\varepsilon)}(t) \\ &\quad + \int_0^t S_\varepsilon(t-s) G(u_\varepsilon(s)) D_\varepsilon u(s) ds. \end{aligned} \quad (2.10)$$

Note that for fixed $\varepsilon > 0$, we do not need rough path theory to solve this fixed point problem. The existence of local solutions can for example be shown through a fixed point argument in \mathcal{C}_T . For positive ε the approximate derivative operator D_ε is actually continuous on the space of continuous functions, but the operator norm blows up as ε goes to zero. Therefore, it will be useful to introduce approximate reference rough paths $(X_\varepsilon, \mathbf{X}_\varepsilon)$ and interpret the term involving G on the right-hand side of (2.10) as an approximation to a rough integral.

We choose X_ε , the solution of the approximated stochastic heat equation

$$dX_\varepsilon = \Delta_\varepsilon X_\varepsilon dt + H_\varepsilon dW(t),$$

as a reference rough path. For convenience we assume zero initial conditions for X_ε so that we get

$$X_\varepsilon(t) = \int_0^t S_\varepsilon(t-s) H_\varepsilon dW(s).$$

Our first task then consists of checking that for every s the process $X_\varepsilon(s)$ can indeed be lifted to a rough path $(X_\varepsilon(s), \mathbf{X}_\varepsilon(s))$ and that for a given adapted process θ the approximate stochastic convolutions Ψ_ε^θ are indeed controlled by X_ε . This is established in Section 3. To be more precise, we will give bounds on the Hölder regularity of Ψ^θ in Lemma 3.1. The behaviour of $(X_\varepsilon, \mathbf{X}_\varepsilon)$ as a rough path is discussed in Lemma 3.3. Finally, in Lemma 3.6 the regularity of the remainder

$$R_\varepsilon^\theta(t; x, y) = (\Psi_\varepsilon^\theta(t, y) - \Psi_\varepsilon^\theta(t, x)) - \theta(t, x)(X_\varepsilon(t, y) - X_\varepsilon(t, x)) \quad (2.11)$$

is treated. For all of these quantities we can show convergence as ε goes to zero towards the corresponding terms for the limiting equation.

Let us point out that, while the derivations of the a priori bounds on Ψ_ε^θ and on $(X_\varepsilon, \mathbf{X}_\varepsilon)$ are rather straightforward, the result for the remainder R_ε^θ requires more thought. There are two key difficulties. The first one lies in getting bounds on the integrals

$$\int_0^t \int_{-\pi}^\pi (p_{t-s}^\varepsilon(x_1 - y) - p_{t-s}^\varepsilon(x_2 - y))^2 |x_1 - y|^{2\alpha} dy ds. \quad (2.12)$$

In [HW10] the corresponding bounds were obtained in the limiting case $\varepsilon = 0$ using the representation

$$p_t(x) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{4\pi t}} e^{\frac{(x-2\pi k)^2}{4t}},$$

which is not available for $\varepsilon > 0$. We circumvent this problem by adapting a result of Stein [Ste57] on Fourier multipliers on weighted spaces to the current context. These results can be found in Lemma 6.1 and Lemma 6.2.

The second main problem lies in the lack of time regularity of the approximated heat semigroup S_ε near 0. As pointed out in (2.6) in the case $\varepsilon = 0$ the *space-time* regularity of u is needed to show the *spatial* regularity of $R^{\theta(u)}$. In the case $\varepsilon > 0$ we establish in Lemma 6.6 that $S_\varepsilon(t)$ does have the necessary time regularity for times $t \geq \varepsilon^2$. To this end we make use of the Markiewicz multiplier theorem. We use this to show that the remainder R_ε^θ does have the desired spatial regularity for times $t \geq \varepsilon^2$ with a blowup for $t \downarrow \varepsilon^2$.

Once we have established that Ψ_ε^θ is well behaved, it remains to check the behaviour of the term

$$\int_0^t \left[\int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) G(u_\varepsilon(s, y)) D_\varepsilon u_\varepsilon(s, y) dy \right] ds \quad (2.13)$$

when $\varepsilon \downarrow 0$. One might hope that for small ε the integral in square brackets behaves like an approximation to the rough integral

$$\int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) G(u_\varepsilon(s, y)) d_y u_\varepsilon(s, y). \quad (2.14)$$

Unfortunately, this is not true. As pointed out in Appendix A, rough integrals are limits of second order Riemann sums like (A.5). Since the contribution of the second order term may not be negligible in the limit, one cannot hope to prove that the first order expression in the second line of (2.13) approximates the rough integral (2.14) in general. In order to enforce this convergence, we simply add the “missing” second order term to the right-hand side of (2.13).

Therefore, we set

$$\begin{aligned} \Xi_\varepsilon^{u_\varepsilon}(t, \cdot) &:= \int_0^t \left[\int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) G(u_\varepsilon(s, y)) D_\varepsilon u_\varepsilon(s, y) dy \right] ds \\ &+ \int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) DG(u_\varepsilon(s, y)) u'_\varepsilon(s, y) D_\varepsilon \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) dy ds, \end{aligned} \quad (2.15)$$

where

$$D_\varepsilon \mathbf{X}_\varepsilon(s, y) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \mathbf{X}_\varepsilon(s, y, y + \varepsilon z) \mu(dz),$$

and u'_ε is a *rough path derivative* of u_ε with respect to X_ε .

We will denote the extra term appearing on the right-hand side of (2.15) by

$$\begin{aligned} \Upsilon_\varepsilon^{u_\varepsilon}(t, \cdot) &:= \int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) DG(u_\varepsilon(s, y)) \\ &\times u'_\varepsilon(s, y) D_\varepsilon \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) dy ds. \end{aligned} \quad (2.16)$$

Actually, here we have hidden a bit of non-trivial linear algebra in the notation. The expression defining $\Upsilon_\varepsilon^{u_\varepsilon}$ is trilinear and it is not obvious which term is paired with which. At this level, this does not matter and we will give a precise definition in (5.4) below.

In Section 5 we establish that, under suitable assumptions, $\Xi_\varepsilon^{u_\varepsilon}(t)$ approximates

$$\Xi^u(t) := \int_0^t \left[\int_{-\pi}^\pi p_{t-s}(\cdot - y) G(u(s, y)) d_y u(s, y) \right] ds,$$

provided that the quantity

$$\mathcal{D}_\varepsilon = \left[\|X - X_\varepsilon\|_{C_T^\alpha} + \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}_T^{2\alpha}} + \|u - u_\varepsilon\|_{C_T^\alpha} + \|u' - u'_\varepsilon\|_{C_{[\varepsilon^2, T]}^\alpha} \right]$$

$$\left. + \|R_u - R_{u_\varepsilon}\|_{\mathcal{B}_{[\varepsilon^2, T], \beta}^{2\alpha}} \right]$$

is small. A key ingredient of the proof of this statement is Lemma 5.2 which treats the behaviour of rescaled rough integrals under approximation. As above the approximation of the heat semigroup poses an obstacle. In Lemma 6.4 we refer to the Marcinkiewicz multiplier theorem to prove that it satisfies the right regularity properties.

By adding a rough path derivative and a remainder we can then interpret u_ε as solution to the fixed point problem for the operator

$$(u_\varepsilon, u'_\varepsilon, R_{u_\varepsilon}) \mapsto \mathcal{M}_\varepsilon(u_\varepsilon, u'_\varepsilon, R_{u_\varepsilon}) = (\tilde{u}_\varepsilon, \tilde{u}'_\varepsilon, R_{\tilde{u}_\varepsilon})$$

where

$$\begin{aligned} \tilde{u}_\varepsilon(t) &:= \left(S_\varepsilon u_\varepsilon^0 + \Psi_\varepsilon^{\theta(u_\varepsilon)} + \Phi_\varepsilon^{F(u_\varepsilon)} - \Upsilon_\varepsilon^{u_\varepsilon} + \Xi_\varepsilon^{u_\varepsilon} \right)(t \wedge \tau) \\ \tilde{u}'_\varepsilon(t) &:= \theta(u_\varepsilon(t \wedge \tau)), \\ R_{\tilde{u}_\varepsilon}(t) &:= \delta \left(S_\varepsilon u_\varepsilon^0 + \Phi_\varepsilon^{F(u_\varepsilon)} - \Upsilon_\varepsilon^{u_\varepsilon} + \Xi_\varepsilon^{u_\varepsilon} \right)(t \wedge \tau) \\ &\quad + \left(\delta \Psi_\varepsilon^{\theta(u_\varepsilon)} - \theta(u_\varepsilon) \delta X_\varepsilon \right)(t \wedge \tau), \end{aligned} \quad (2.17)$$

for a suitable stopping time τ . Here the operator δ is defined as in (2.8).

As expected, the term $\Upsilon_\varepsilon^{u_\varepsilon}$ will be responsible for the emergence of an extra term in the limit. To treat it in Section 4 we will discuss the convergence of the term $D_\varepsilon \mathbf{X}_\varepsilon$.

It turns out that for ε small enough this term behaves like ΛId , where Λ is the constant introduced above in (1.12). Note that the a priori knowledge of the regularity of \mathbf{X}_ε would not even imply that the quantity $\Upsilon_\varepsilon^{u_\varepsilon}$ remains bounded. The convergence is a consequence of stochastic cancellations. The relevant bound is given in Proposition 4.1. There, convergence in any stochastic L^p space with respect to the Sobolev norm $H^{-\alpha}$ is proved. In the next subsection we will be able to conclude convergence of the triple $(u_\varepsilon, u'_\varepsilon, R_{u_\varepsilon})$.

2.3. Proof of the main result. Now we are ready to finish the proof of our main result, Theorem 1.5, assuming the results from Sections 3 – 6.

Similar to (2.16) we define

$$\Upsilon^{\bar{u}}(t, \cdot) := \Lambda \int_0^t \int_{-\pi}^\pi p_{t-s}(\cdot - y) DG(\bar{u}(s, y)) \bar{u}'(s, y) \bar{u}'(s, y) dy ds \quad (2.18)$$

where the indices are to be interpreted as in (1.11). Then the mild form of (1.10) can be written as

$$\begin{aligned} \bar{u}(t) &:= S(t) u^0 + \Psi^{\theta(\bar{u})}(t) + \Phi^{F(\bar{u})}(t) - \Upsilon^{\bar{u}}(t) + \Xi^{\bar{u}}(t), \\ \bar{u}'(t) &:= \theta(\bar{u}(t)), \\ R_{\bar{u}}(t) &:= \delta \left(S u^0 + \Phi^{F(\bar{u})} - \Upsilon^{\bar{u}} + \Xi^{\bar{u}} \right)(t) + \left(\delta \Psi^{\theta(\bar{u})} - \theta(\bar{u}) \delta X \right)(t). \end{aligned} \quad (2.19)$$

Furthermore, for $t < \tau_\varepsilon^*$ the process u_ε solves the fixed point problem for the operator \mathcal{M}_ε defined in (2.17). Recall that τ^* denotes the explosion time of \bar{u} . Similarly, here τ_ε^* denotes the explosion time for u_ε . Note that the extra term $\Upsilon^{\bar{u}}$ corresponds to a reaction term and poses no additional problems for the well-posedness of the equation.

In order to optimize the convergence rate we have to work with three different Hölder exponents

$$\alpha_\star > \alpha \geq \tilde{\alpha}.$$

Recall that in the introduction we have set $\alpha_\star = \frac{1}{2} - \kappa$ for a sufficiently small $\kappa > 0$. We will fix $\alpha = \alpha_\star - \kappa$ to be a bit smaller and $\tilde{\alpha} \in (\frac{1}{3}, \alpha]$ arbitrary. We will also fix the blow-up rates $\beta = \alpha + \kappa/3$ and $\tilde{\beta} = \tilde{\alpha} + \kappa/3$. Note that the regularising property of the heat semigroup implies that

$$t^{\frac{\beta}{2}} |S(t)u^0|_{2\alpha} \lesssim |u^0|_\alpha \quad \text{and} \quad t^{\frac{\tilde{\beta}}{2}} |S_\varepsilon(t)u^0|_{2\alpha} \lesssim |u^0|_\alpha,$$

and similarly for $\tilde{\alpha}, \tilde{\beta}$. Actually, in the case of the heat semigroup S this is a standard regularity result and it would even be true for $\beta = \alpha$. For the approximated semigroup S_ε the regularisation is shown in Lemma 6.5.

The reason for introducing these three different exponents is the following. The convergence rates of some of the quantities considered in Section 3 become better when measured in a lower regularity norm. But the bounds on $|D_\varepsilon \mathbf{X} - \Lambda \text{Id}|_{H^{-\alpha}}$ in Proposition 4.1 as well as the convergence rates of the approximated rough integrals (see e.g. Lemma 5.2) become better for larger spatial regularity. Hence it is useful to use a priori knowledge on the boundedness of the α norms for these terms. The α_\star is only needed to bound the deterministic part of the equation. It accounts for the loss of regularity in the bounds on the approximate heat semigroup.

First we introduce the following stopping times. Recall the definitions of the norms $\|\cdot\|_{C_t^{\alpha/2, \alpha}}, \|\cdot\|_{C_t^\alpha}$ from (1.15) and (1.16). Then for any $K > 0$ we define:

$$\sigma_K^X := \inf \left\{ t: \|X\|_{C_t^{\alpha/2, \alpha}} \geq K \quad \text{or} \quad \|\mathbf{X}\|_{\mathcal{B}_t^{2\alpha}} \geq K \right\}, \quad (2.20a)$$

$$\sigma_K^u := \inf \left\{ t: |\bar{u}(t)|_{C^{\alpha_\star}} \geq K \quad \text{or} \quad \|\bar{u}\|_{C_t^{\alpha/2, \alpha}} \geq K \right\}, \quad (2.20b)$$

$$\sigma_K^R := \inf \left\{ t: |R_{\bar{u}}(t)|_{2\alpha} \geq Kt^{-\frac{\beta}{2}} \quad \text{or} \quad |R_{\bar{u}}(t)|_{2\tilde{\alpha}} \geq Kt^{-\frac{\tilde{\beta}}{2}} \right\}. \quad (2.20c)$$

Here we follow the convention to set the stopping times to be T if the sets are empty. It follows from the bounds in Sections 3 – 5 that for suitable initial condition u^0 these stopping times are almost surely positive. Remark that for $t \leq \sigma_K^u$ we have

$$\|\bar{u}'\|_{C_t^\alpha} = \|\theta(\bar{u})\|_{C_t^\alpha} \leq \sup_{|u| \leq K} |\theta(u)| + \sup_{|u| \leq K} |D\theta(u)|K. \quad (2.21)$$

Then we set

$$\sigma_K = \sigma_K^X \wedge \sigma_K^u \wedge \sigma_K^R.$$

In order to have a priori bounds on the corresponding ε -quantities we introduce the following stopping times

$$\begin{aligned} \varrho_\varepsilon^X := \inf \left\{ t: \|X - X_\varepsilon\|_{C_t^{\alpha/2, \alpha}} \geq 1, \quad \text{or} \quad \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}_t^{2\alpha}} \geq 1, \right. \\ \left. \text{or} \quad \left| D_\varepsilon \mathbf{X}_\varepsilon(t) - \Lambda_\varepsilon(t) \text{Id} \right|_{H^{-\alpha}} \geq 1 \right\}, \end{aligned} \quad (2.22a)$$

$$\varrho_\varepsilon^u := \inf \left\{ t: \|\bar{u} - u_\varepsilon\|_{C_t^\alpha} \geq 1 \right\} \wedge \inf \left\{ t > \varepsilon^2: \|\bar{u} - u_\varepsilon\|_{C_{[\varepsilon^2, t]}^{\alpha/2, \alpha}} \geq 1 \right\}, \quad (2.22b)$$

$$\varrho_\varepsilon^R := \inf \left\{ t > \varepsilon^2: |R_{\bar{u}}(t) - R_{u_\varepsilon}(t)|_{2\alpha} \geq (t - \varepsilon^2)^{-\frac{\beta}{2}} \right\}$$

$$\text{or } |R_{\bar{u}}(t) - R_{u_\varepsilon}(t)|_{2\tilde{\alpha}} \geq (t - \varepsilon^2)^{-\frac{\tilde{\beta}}{2}} \Big\}. \quad (2.22c)$$

The auxiliary constant $\Lambda_\varepsilon(t)$ is a natural approximation to Λ which is defined in (4.2) below. In (2.22c) the quantity R_{u_ε} is defined as in (2.17). Finally, define

$$\varrho_{K,\varepsilon} = \sigma_K \wedge \varrho_\varepsilon^X \wedge \varrho_\varepsilon^u \wedge \varrho_\varepsilon^R,$$

where again, we set the stopping times equal to T if the sets are empty. It is clear from the definition that for $t \leq \varrho_{K,\varepsilon}$ we have deterministic bounds on

$$\begin{aligned} & \|X_\varepsilon\|_{\mathcal{C}_t^{\alpha/2,\alpha}}, \|X_\varepsilon\|_{\mathcal{B}_t^{2\alpha}}, |D_\varepsilon X_\varepsilon(t)|_{H^{-\alpha}}, \|u_\varepsilon\|_{\mathcal{C}_t^\alpha}, \|u_\varepsilon\|_{\mathcal{C}_{[\varepsilon^2,t]}^{\alpha/2,\alpha}}, \\ & |R_{u_\varepsilon}|_{\mathcal{B}_{[\varepsilon^2,t],\beta}^{2\alpha}}, |R_{u_\varepsilon}|_{\mathcal{B}_{[\varepsilon^2,t],\tilde{\beta}}^{2\tilde{\alpha}}}, \|u'_\varepsilon\|_{\mathcal{C}_t^\alpha} = \|\theta(u_\varepsilon)\|_{\mathcal{C}_t^\alpha}. \end{aligned}$$

From now on, to reduce the number of indices, we will write

$$t_\varepsilon := t \wedge \varrho_{K,\varepsilon}.$$

Most of the rest of this subsection will be devoted to the proof of the following theorem which implies our main result, Theorem 1.5.

Theorem 2.3. *Let the exponents $\alpha_*, \alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ be as stated at the beginning of this subsection. Suppose that the initial conditions satisfy*

$$|u^0|_{\mathcal{C}^{\alpha_*}} < K \quad \text{and} \quad |u_\varepsilon^0|_{\mathcal{C}^{\alpha_*}} < K + 1$$

for some large constant K . Then there exists a constant

$$\gamma = \gamma(\tilde{\alpha}, \kappa) > 0,$$

such that if

$$|u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}} \lesssim \varepsilon^\gamma,$$

then for any terminal time $T > 0$ we have for any $\tilde{\gamma} < \gamma$

$$\mathbb{P} \left[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{T_\varepsilon}^{\tilde{\alpha}}} + \|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{[\varepsilon^2, T_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}} + \|R_{\bar{u}} - R_{u_\varepsilon}\|_{\mathcal{B}_{[\varepsilon^2, T_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}} \geq \varepsilon^{\tilde{\gamma}} \right] \rightarrow 0. \quad (2.23)$$

Furthermore, for every fixed $\tilde{\alpha}$, the constant $\gamma(\tilde{\alpha}, \kappa)$ can be chosen arbitrarily close to $\frac{1}{2} - \tilde{\alpha}$ by taking $\kappa > 0$ sufficiently small.

Proof of Theorem 2.3. Throughout the argument, we will always be dealing with times $t \leq \varrho_{K,\varepsilon}$ and the functions F, G, θ will only be evaluated for u with $|u| \leq K + 1$. All the quantities of interest will remain unchanged if we change F, G and θ outside a ball. Therefore, from now on we can and will make the additional assumption that

$$|F|_{\mathcal{C}^1} < \infty \quad |G|_{\mathcal{C}^3} < \infty, \quad |\theta|_{\mathcal{C}^2} < \infty.$$

For any $\tilde{\alpha} \in (\frac{1}{3}, \alpha]$ we will derive a bound on the quantity

$$\mathcal{E}^{\tilde{\alpha}}(t) := \mathbb{E} \left[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|R_{\bar{u}} - R_{u_\varepsilon}\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}}.$$

Using the equations (2.19) and (2.17) that \bar{u} and u_ε satisfy we get the bound

$$\mathcal{E}^{\tilde{\alpha}}(t) \lesssim \sum_{i=1}^5 I_i^{\tilde{\alpha}}(t), \quad (2.24)$$

where

$$I_1^{\tilde{\alpha}}(t) := \|S(\cdot)u^0 - S_\varepsilon(\cdot)u_\varepsilon^0\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}} + \|S(\cdot)u^0 - S_\varepsilon(\cdot)u_\varepsilon^0\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}} \\ + \sup_{\varepsilon^2 < s \leq t_\varepsilon} (s - \varepsilon^2)^{\frac{\tilde{\beta}}{2}} |S(s)u^0 - S_\varepsilon(s)u_\varepsilon^0|_{\mathcal{C}^{2\tilde{\alpha}}}, \quad (2.25a)$$

$$I_2^{\tilde{\alpha}}(t) := \mathbb{E} \left[\|\Psi^{\theta(\bar{u})} - \Psi_\varepsilon^{\theta(u_\varepsilon)}\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ + \mathbb{E} \left[\|R^{\theta(\bar{u})} - R_\varepsilon^{\theta(u_\varepsilon)}\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}}, \quad (2.25b)$$

$$I_3^{\tilde{\alpha}}(t) := \mathbb{E} \left[\|\Phi^{F(\bar{u})} - \Phi_\varepsilon^{F(u_\varepsilon)}\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ + \mathbb{E} \left[\left(\sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} |\Phi^{F(\bar{u})}(s) - \Phi_\varepsilon^{F(u_\varepsilon)}(s)|_{\mathcal{C}^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}}, \quad (2.25c)$$

$$I_4^{\tilde{\alpha}}(t) := \mathbb{E} \left[\|\Upsilon^{\bar{u}} - \Upsilon_\varepsilon^{u_\varepsilon}\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ + \mathbb{E} \left[\left(\sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} |\Upsilon^{\bar{u}}(s) - \Upsilon_\varepsilon^{u_\varepsilon}(s)|_{\mathcal{C}^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}}, \quad (2.25d)$$

$$I_5^{\tilde{\alpha}}(t) := \mathbb{E} \left[\|\Xi^{\bar{u}} - \Xi_\varepsilon^{u_\varepsilon}\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|\Xi^{\bar{u}} - \Xi_\varepsilon^{u_\varepsilon}\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ + \mathbb{E} \left[\left(\sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} |\Xi^{\bar{u}}(s) - \Xi_\varepsilon^{u_\varepsilon}(s)|_{\mathcal{C}^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}}. \quad (2.25e)$$

Actually, in $I_1^{\tilde{\alpha}} - I_5^{\tilde{\alpha}}$ we give slightly more information than needed. Note in particular, that in $I_3^{\tilde{\alpha}}, I_4^{\tilde{\alpha}}$, and $I_5^{\tilde{\alpha}}$ we allow for blowup at 0 not at ε^2 . This bound is strictly stronger.

We start by giving a bound on $I_1^{\tilde{\alpha}}$. For every $t > \varepsilon^2$ we get

$$\|S(\cdot)u^0 - S_\varepsilon(\cdot)u_\varepsilon^0\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}} \\ \leq \|S(\cdot)(u^0 - u_\varepsilon^0)\|_{\mathcal{C}_t^{\tilde{\alpha}/2, \tilde{\alpha}}} + \|(S(\cdot) - S_\varepsilon(\cdot))u_\varepsilon^0\|_{\mathcal{C}_{[\varepsilon^2, t]}^{\tilde{\alpha}/2, \tilde{\alpha}}} \quad (2.26) \\ \lesssim |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}} + |u_\varepsilon^0|_{\mathcal{C}^{\alpha_*}} \varepsilon^{\alpha_* - \tilde{\alpha} - \kappa/2}.$$

Here we have used the fact that the heat semigroup is a contraction from \mathcal{C}^α to \mathcal{C}^α as well as the time continuity of the heat semigroup in the first term. In the second term we have used Lemma 6.4 and Lemma 6.5, which provide bounds on the spatial regularisation due to the approximated heat semigroup. We use Lemma 6.6 to get the temporal regularity. Let us stress again that this lemma only yields information for times $t > \varepsilon^2$.

The remaining terms in (2.25a),

$$\|S(\cdot)u^0 - S_\varepsilon(\cdot)u_\varepsilon^0\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}} \quad \text{and} \quad \sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} |S(s)u^0 - S_\varepsilon(s)u_\varepsilon^0|_{\mathcal{C}^{2\tilde{\alpha}}}, \quad (2.27)$$

can be bounded by the same quantity. Here we use that the Lemmas 6.4 and 6.5 about the spatial regularity hold for arbitrary times. We also use the fact that $\tilde{\beta} = \tilde{\alpha} + \kappa/3$ has been chosen in order to compensate for the blowup of the $\mathcal{C}^{2\tilde{\alpha}}$ -norm

of $S(t)u^0$ near zero. Hence, we can conclude that

$$I_1^{\tilde{\alpha}}(t) \lesssim |u^0 - u_\varepsilon^0|_{C^{\tilde{\alpha}}} + \varepsilon^{\alpha_* - \tilde{\alpha} - \kappa/2}. \quad (2.28)$$

This is the only part in the argument, where we will use the boundedness of u_ε^0 or u^0 in the C^{α_*} -norm.

The bounds on $I_2^{\tilde{\alpha}}$ are derived in Section 3. More specifically, for

$$p > \frac{6}{1 - 2\tilde{\alpha}} \quad \text{and} \quad \lambda_1 = 1 - 2\tilde{\alpha} - \frac{6}{p}$$

we get using Corollary 3.2

$$\begin{aligned} & \mathbb{E} \left[\left\| \Psi^{\theta(\bar{u})} - \Psi_\varepsilon^{\theta(u_\varepsilon)} \right\|_{C_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ & \leq \mathbb{E} \left[\left\| \Psi^{\theta(\bar{u})} - \Psi^{\theta(u_\varepsilon)} \right\|_{C_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\left\| \Psi^{\theta(u_\varepsilon)} - \Psi_\varepsilon^{\theta(u_\varepsilon)} \right\|_{C_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ & \lesssim t^{\frac{\lambda_1}{4}} \mathbb{E} \left[\left\| \theta(\bar{u}) - \theta(u_\varepsilon) \right\|_{C_{t_\varepsilon}}^p \right]^{\frac{1}{p}} + \varepsilon^{\lambda_1 \alpha} \mathbb{E} \left[\left\| \theta(u_\varepsilon) \right\|_{C_{t_\varepsilon}^\alpha}^p \right]^{\frac{1}{p}} \\ & \lesssim t^{\frac{\lambda_1}{4}} \mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{C_{t_\varepsilon}}^p \right]^{\frac{1}{p}} + \varepsilon^{\lambda_1 \alpha} \mathbb{E} \left[\left\| u_\varepsilon \right\|_{C_{t_\varepsilon}^\alpha}^p \right]^{\frac{1}{p}} \\ & \lesssim t^{\frac{\lambda_1}{4}} \mathcal{E}^{\tilde{\alpha}}(t) + \varepsilon^{\lambda_1 \alpha} (K + 1). \end{aligned} \quad (2.29)$$

In passing from the second to the third line, we have used (3.14) and (3.15) as well as the linearity of the map $\theta \mapsto \Psi^\theta$. When passing from the third to the fourth line, we have used the definition of the stopping times $\varrho_{K,\varepsilon}$ and in particular the fact that for $t \leq \varrho_{K,\varepsilon}$ the C^1 -norm of θ is bounded by a deterministic constant.

In particular, by choosing p large enough and κ small enough the rate $\lambda_1 \alpha$ can be increased arbitrarily close to $\frac{1}{2} - \tilde{\alpha}$.

In order to get a bound on the second quantity in $I_2^{\tilde{\alpha}}$ we write

$$\begin{aligned} & \mathbb{E} \left[\left\| R^{\theta(\bar{u})} - R_\varepsilon^{\theta(u_\varepsilon)} \right\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ & \leq \mathbb{E} \left[\left\| R^{\theta(\bar{u})} - R_\varepsilon^{\theta(\bar{u})} \right\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\left\| R_\varepsilon^{\theta(\bar{u})} - R_\varepsilon^{\theta(u_\varepsilon)} \right\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}}. \end{aligned} \quad (2.30)$$

The first term on the right hand side of (2.30) can be bounded directly using Corollary 3.7. Actually, using the time regularity of $\theta(u)$ for all times in $[0, \varrho_{K,\varepsilon}]$ we even get a bound without blowup. Then for

$$p > \frac{2 + 6\alpha}{\alpha(1 + 2\alpha - 4\tilde{\alpha})} \quad \text{and} \quad \lambda_2 = 1 - \frac{4\tilde{\alpha}}{1 + 2\alpha} - \frac{1}{p} \frac{2 + 6\alpha}{\alpha(1 + 2\alpha)}$$

we obtain

$$\begin{aligned} \mathbb{E} \left[\left\| R^{\theta(\bar{u})} - R_\varepsilon^{\theta(\bar{u})} \right\|_{\mathcal{B}_{t_\varepsilon}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} & \lesssim \varepsilon^{\lambda_2 \alpha} \mathbb{E} \left[\left\| \theta(\bar{u}) \right\|_{C_{t_\varepsilon}^{\alpha/2, \alpha}}^p \right]^{\frac{1}{p}} \\ & \lesssim \varepsilon^{\lambda_2 \alpha} \mathbb{E} \left[\left\| \bar{u} \right\|_{C_{t_\varepsilon}^{\alpha/2, \alpha}}^p \right]^{\frac{1}{p}} \lesssim K \varepsilon^{\lambda_2 \alpha}. \end{aligned}$$

Here we have used the fact that the stopping time $\varrho_{K,\varepsilon}$ is almost surely smaller than the stopping time $\varrho_{\varepsilon,K}^X$ defined in (3.49). Note that as above by choosing κ small enough and p large enough the rate $\lambda_2 \alpha$ can again be increased arbitrarily close to $\frac{1}{2} - \tilde{\alpha}$.

The bound on the remaining term in (2.30) requires more thought. Here we write

$$R_\varepsilon^{\theta(\bar{u})} - R_\varepsilon^{\theta(u_\varepsilon)} = R_\varepsilon^{\theta(\bar{u}) - \theta(u_\varepsilon)} = R_{(\varepsilon^2)^+, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)} + R_{(\varepsilon^2)^-, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)}.$$

The processes $R_{(\varepsilon^2)^\pm, \varepsilon}^{\theta}$ are defined below in (3.47) and (3.48). For $R_{(\varepsilon^2)^+, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)}$ we can use Corollary 3.7 once more to obtain for

$$p > \frac{2 + 6\tilde{\alpha}}{\alpha(1 - 2\tilde{\alpha})} \quad \text{and} \quad \lambda_3 = 1 - \frac{4\tilde{\alpha}}{1 + 2\tilde{\alpha}} - \frac{1}{p} \frac{2 + 6\tilde{\alpha}}{\tilde{\alpha}(1 + 2\tilde{\alpha})} > 0,$$

that

$$\begin{aligned} \mathbb{E} \left[\left\| R_{(\varepsilon^2)^+, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)} \right\|_{\mathcal{B}_{[\varepsilon^2, t_\varepsilon]}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} &\lesssim t^{\lambda_3 \tilde{\alpha}/2} \mathbb{E} \left[\left\| \theta(\bar{u}) - \theta(u_\varepsilon) \right\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ &\lesssim t^{\lambda_3 \tilde{\alpha}/2} \mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ &\lesssim t^{\lambda_3 \tilde{\alpha}/2} \mathcal{E}^{\tilde{\alpha}}(t). \end{aligned}$$

To bound give a bound on $R_{(\varepsilon^2)^-, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)}$ we refer to Lemma 3.8. We get for $p > \frac{6}{1-2\alpha}$ that

$$\mathbb{E} \left\| R_{(\varepsilon^2)^-, \varepsilon}^{\theta(\bar{u}) - \theta(u_\varepsilon)} \right\|_{\mathcal{B}_{[s, t_\varepsilon]}^{2\tilde{\alpha}}, \tilde{\beta}}^p \lesssim T^{\lambda_4} \mathbb{E} \left\| \theta(\bar{u}) - \theta(u_\varepsilon) \right\|_{\mathcal{C}_\tau}^p \lesssim T^{\lambda_4} \mathcal{E}^{\tilde{\alpha}}(t).$$

where $\lambda_4 = \alpha - \tilde{\alpha} + \kappa/2$.

The bounds on $I_3^{\tilde{\alpha}}$ and $I_4^{\tilde{\alpha}}$ are provided in Section 4. Using Proposition 4.6 twice we get

$$\begin{aligned} I_3^{\tilde{\alpha}}(t) &= \mathbb{E} \left[\left\| \Phi^{F(\bar{u})} - \Phi_\varepsilon^{F(u_\varepsilon)} \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ &\quad + \mathbb{E} \left[\left(\sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} \left| \Phi^{F(\bar{u})}(s) - \Phi_\varepsilon^{F(u_\varepsilon)}(s) \right|_{\mathcal{C}^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}} \\ &\lesssim t^{1 - \frac{\tilde{\alpha}}{2} - \kappa} \mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \varepsilon^{1 - \kappa}. \end{aligned} \quad (2.31)$$

Here we have used that due to the definition before the stopping time $\varrho_{K, \varepsilon}$ the norms $\|\bar{u}\|_{\mathcal{C}^\alpha}$ and $\|u_\varepsilon\|_{\mathcal{C}^\alpha}$ are bounded by K .

Then using Proposition 4.7 we get

$$\begin{aligned} I_4^{\tilde{\alpha}}(t) &= \mathbb{E} \left[\left\| \Upsilon^{\bar{u}} - \Upsilon_\varepsilon^{u_\varepsilon} \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\left(\sup_{0 < s \leq t_\varepsilon} s^{\frac{\tilde{\beta}}{2}} \left| \Upsilon^{\bar{u}} - \Upsilon_\varepsilon^{u_\varepsilon} \right|_{\mathcal{C}^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}} \\ &\lesssim \varepsilon^{1 - \tilde{\alpha} - \kappa} + \sup_{0 < t \leq T} t^{\frac{\tilde{\alpha}}{2}} \left| \Lambda_\varepsilon(t) - \Lambda \right| \\ &\quad + \mathbb{E} \left[\sup_{t \in [0, T]} \left| D_\varepsilon \mathbf{X}_\varepsilon(t, \cdot) - \Lambda_\varepsilon(t) \text{Id} \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} \\ &\quad + t^{\frac{1 + \alpha - \tilde{\alpha}}{2}} \left(\mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\left\| \theta(\bar{u}) - \theta(u_\varepsilon) \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} \right) \\ &\lesssim \varepsilon^{\alpha - \kappa} + t^{\frac{1 + \alpha - \tilde{\alpha}}{2}} \mathcal{E}^{\tilde{\alpha}}(t). \end{aligned}$$

We use Proposition 5.1, the main results of Section 5, to bound $I_5^{\tilde{\alpha}}$. Using (5.6) and (5.7) for $\gamma = \tilde{\alpha}$ and $\tilde{\gamma}$ large enough, we get

$$\begin{aligned} \mathbb{E} \left[\left\| \Xi^{\bar{u}} - \Xi_{\varepsilon}^{u_{\varepsilon}} \right\|_{C_{[\varepsilon^2, t_{\varepsilon}]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\left\| \Xi^{\bar{u}} - \Xi_{\varepsilon}^{u_{\varepsilon}} \right\|_{C_{t_{\varepsilon}}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ \lesssim \mathbb{E} [\mathcal{D}_{\varepsilon}^p]^{\frac{1}{p}} t^{\frac{1-\tilde{\beta}-\kappa}{2}} + \varepsilon^{3\alpha-1} + \varepsilon^{2-3\tilde{\alpha}-\tilde{\beta}}. \end{aligned} \quad (2.32)$$

Using (5.6) again for $\gamma = 2\tilde{\alpha}$ we get

$$\mathbb{E} \left[\left(\sup_{0 < s \leq t_{\varepsilon}} s^{-\frac{\tilde{\beta}}{2}} \left| \Xi^{\bar{u}}(s) - \Xi_{\varepsilon}^{u_{\varepsilon}}(s) \right|_{C^{2\tilde{\alpha}}} \right)^p \right]^{\frac{1}{p}} \lesssim \mathbb{E} [\mathcal{D}_{\varepsilon}^p]^{\frac{1}{p}} t^{\frac{1-\tilde{\beta}-\kappa}{2}} + \varepsilon^{1-2\tilde{\alpha}-\kappa}. \quad (2.33)$$

Note that we have used again, that due to the stopping time $\varrho_{K,\varepsilon}$ all the relevant norms are bounded. In particular, the constants that are suppressed in the \lesssim notation depend on K . As above, in (2.32), (2.33) we have used the notation

$$\begin{aligned} \mathcal{D}_{\varepsilon} = & \|X - X_{\varepsilon}\|_{C_T^{\tilde{\alpha}}} + \|\mathbf{X} - \mathbf{X}_{\varepsilon}\|_{\mathcal{B}_T^{2\tilde{\alpha}}} + \|u - u_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\tilde{\alpha}}} + \|\theta(u) - \theta(u_{\varepsilon})\|_{C_{[\varepsilon^2, t_{\varepsilon}]}^{\tilde{\alpha}}} \\ & + \|R_u - R_{u_{\varepsilon}}\|_{\mathcal{B}_{[\varepsilon^2, t_{\varepsilon}], \tilde{\beta}/2}^{2\tilde{\alpha}}}. \end{aligned}$$

The quantity $\mathbb{E} [\mathcal{D}_{\varepsilon}^p]^{\frac{1}{p}}$ can be bounded by

$$\begin{aligned} \mathbb{E} [\mathcal{D}_{\varepsilon}^p]^{\frac{1}{p}} & \lesssim \mathbb{E} \left[\|X - X_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|\mathbf{X} - \mathbf{X}_{\varepsilon}\|_{\mathcal{B}_{t_{\varepsilon}}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|\bar{u} - u_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ & + \mathbb{E} \left[\|R_u - R_{u_{\varepsilon}}\|_{\mathcal{B}_{[\varepsilon^2, t_{\varepsilon}], \tilde{\beta}/2}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} \\ & \lesssim \mathbb{E} \left[\|X - X_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathbb{E} \left[\|\mathbf{X} - \mathbf{X}_{\varepsilon}\|_{\mathcal{B}_{t_{\varepsilon}}^{2\tilde{\alpha}}}^p \right]^{\frac{1}{p}} + \mathcal{E}^{\tilde{\alpha}}(t). \end{aligned} \quad (2.34)$$

Here we have used the fact that $\bar{u}' = \theta(\bar{u})$ and $u'_{\varepsilon} = \theta(u_{\varepsilon})$ as well as the bound

$$\|\theta(\bar{u}) - \theta(u_{\varepsilon})\|_{C_{[\varepsilon^2, t_{\varepsilon}]}^{\alpha}} \leq |\theta|_{C^1} \|\bar{u} - u_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\alpha}} + |\theta|_{C^2} \|\bar{u}\|_{C_{t_{\varepsilon}}^{\alpha}} \|\bar{u} - u_{\varepsilon}\|_{C_{t_{\varepsilon}}^{\alpha}}.$$

For the rough paths \mathbf{X}_{ε} and \mathbf{X} we get from Corollary 3.4 that for

$$\lambda_5 < 1 - 2\tilde{\alpha}$$

we get

$$\mathbb{E} \left[\|X_{\varepsilon} - X\|_{C_{t_{\varepsilon}}^{\alpha/2, \alpha}}^p \right]^{\frac{1}{p}} \lesssim \varepsilon^{\frac{\lambda_5}{2}} \quad \text{and} \quad \mathbb{E} \left[\|\mathbf{X}_{\varepsilon} - \mathbf{X}\|_{\mathcal{B}_{t_{\varepsilon}}^{2\tilde{\alpha}}}^p \right]^{1/p} \lesssim \varepsilon^{\frac{\lambda_5}{2}}. \quad (2.35)$$

So finally, we get from (2.24), (2.25), (2.28), (2.29), (2.30), (2.32), (2.33), (2.35)

$$\mathcal{E}^{\tilde{\alpha}}(t) \lesssim t^{\tilde{\gamma}} \mathcal{E}^{\tilde{\alpha}}(t) + \varepsilon^{\gamma} + |u^0 - u_{\varepsilon}^0|_{C^{\tilde{\alpha}}}. \quad (2.36)$$

Here the exponents $\tilde{\gamma}, \gamma > 0$ are the minima of the corresponding exponents in the above calculations. Note that a priori γ depends on κ and p . As the bounds only become better for larger p which in turn implies the bound for smaller p due to Young's inequality we will ignore this dependence. In particular, $\gamma = \gamma(\tilde{\alpha}, \kappa)$ increases to $\frac{1}{2} - \tilde{\alpha}$ as $\kappa \downarrow 0$.

By choosing $t = t_*$ small enough the first term on the right-hand side of (2.36) can be absorbed into the left-hand side. Then we get for some constant K_*

$$\sup_{0 \leq s \leq t_*} \mathcal{E}^{\tilde{\alpha}}(t) \leq K_* \varepsilon^{\gamma} + K_* |u^0 - u_{\varepsilon}^0|_{C^{\tilde{\alpha}}}. \quad (2.37)$$

Now if $t_* \geq \varrho_{K,\varepsilon}$ we are done.

Otherwise, we have to iterate this argument taking $\bar{u}(t_*)$ and $u_\varepsilon(t_*)$ as new initial data. Note that K_* and t_* depend only on the choice of K and $\alpha, \tilde{\alpha}$ but not on ε or the particular choice of initial data as long as they satisfy

$$|u^0|_{\mathcal{C}^{\alpha_*}} < K \quad \text{and} \quad |u_\varepsilon^0|_{\mathcal{C}^{\alpha_*}} < K + 1.$$

The definition of the stopping time σ_K^u in (2.20b) ensures that if $t_* > \varrho_{K,\varepsilon}$ the function $\bar{u}(t_*)$ satisfies this bound. Unfortunately, we do not have an a priori bound on the \mathcal{C}^{α_*} norm of $u_\varepsilon(t_* \wedge \varrho_{K,\varepsilon})$ at our disposal (but only on the \mathcal{C}^α norm!). We resolve this problem for the moment by restricting ourselves to the set

$$\mathfrak{X}_{t_*} := \{|\bar{u} - u_\varepsilon|_{\mathcal{C}^{\alpha_*}} \leq 1\}. \quad (2.38)$$

On this set, by definition we have that $|u_\varepsilon|_{\mathcal{C}^{\alpha_*}} < K + 1$ which is enough to iterate the bound.

Actually, one still has to be a bit careful. By restarting at t_* one obtains the estimate

$$\begin{aligned} & \mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[t_*+\varepsilon^2, 2t_*]}^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \mathbf{1}_{\mathfrak{X}_{t_*}} \right]^{1/p} + \mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[t_*, 2t_*]}^{\tilde{\alpha}}}^p \mathbf{1}_{\mathfrak{X}_{t_*}} \right]^{1/p} \\ & + \mathbb{E} \left[\left\| \widehat{R}_{\bar{u}}(s) - \widehat{R}_{u_\varepsilon}(s) \right\|_{\mathcal{B}_{[t_*+\varepsilon^2, 2t_*]}^{2\tilde{\alpha}}]^{\tilde{\beta}/2}}^p \mathbf{1}_{\mathfrak{X}_{t_*}} \right]^{1/p} \\ & \leq K_* \varepsilon^\gamma + K_* \mathbb{E} \left[|\bar{u}(t_* \wedge \varrho_{K,\varepsilon}) - u_\varepsilon(t_* \wedge \varrho_{K,\varepsilon})|_{\mathcal{C}^{\tilde{\alpha}}}^p \mathbf{1}_{\mathfrak{X}_{t_*}} \right]^{1/p} \\ & \leq (K_* + K_*^2) \varepsilon^\gamma + K_*^2 |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}}. \end{aligned}$$

There are two problems to address: The first problem is that the remainders $\widehat{R}_{\bar{u}}$ and $\widehat{R}_{u_\varepsilon}$ that appear here, are the remainders with respect to the solutions \widehat{X} and \widehat{X}_ε of stochastic heat equation and the approximated stochastic heat equation restarted at t_* . This problem can be resolved easily by observing that according to the definitions (2.17) and (2.19)

$$\begin{aligned} & |(\widehat{R}_{\bar{u}}(s) - \widehat{R}_{u_\varepsilon}(s)) - (R_{\bar{u}}(s) - R_{u_\varepsilon}(s))|_{2\tilde{\alpha}} \\ & \leq |\theta(\bar{u}) - \theta(u_\varepsilon)|_{\mathcal{C}} |X - \widehat{X}|_{\mathcal{C}^{2\tilde{\alpha}}} \\ & \quad + |\theta(\bar{u})|_{\mathcal{C}} |(X(s) - \widehat{X}(s)) - (X_\varepsilon(s) - \widehat{X}_\varepsilon(s))|_{\mathcal{C}^{2\tilde{\alpha}}}. \end{aligned}$$

Observing that

$$X(s) - \widehat{X}(s) = S(s - t_*) X(t_*) \quad \text{and} \quad X_\varepsilon(s) - \widehat{X}_\varepsilon(s) = S_\varepsilon(s - t_*) X_\varepsilon(t_*),$$

and using again the a priori bound in Lemma 3.3 on the moments of $|X|_\alpha$ as well as the regularising property of the heat semigroup and the approximated heat semigroup in Lemma 6.4 we get

$$\begin{aligned} & |(\widehat{R}_{\bar{u}}(s) - \widehat{R}_{u_\varepsilon}(s)) - (R_{\bar{u}}(s) - R_{u_\varepsilon}(s))|_{2\tilde{\alpha}} \\ & \lesssim (s - t_*)^{-\frac{\tilde{\beta}}{2}} |\bar{u}(s) - u_\varepsilon(s)|_{\mathcal{C}} + (s - t_*)^{-\frac{\tilde{\beta}}{2}} \varepsilon^{\frac{\lambda_{\tilde{\alpha}}}{2}}. \end{aligned}$$

The first term has already been bounded and hence - possibly by replacing K_* by $2K_*$ we obtain the bound

$$\mathbb{E} \left[\left\| R_{\bar{u}}(s) - R_{u_\varepsilon}(s) \right\|_{\mathcal{B}_{[t_*+\varepsilon^2, 2t_*]}^{2\tilde{\alpha}}]^{\tilde{\beta}/2}}^p \mathbf{1}_{\mathfrak{X}_{t_*}} \right]^{1/p} \leq (K_* + K_*^2) \varepsilon^\gamma + K_*^2 |u^0 - u_\varepsilon^0|.$$

The second issue concerns the weight $(s - t_*)^{\tilde{\beta}}$. A priori this bound does not imply the convergence of $\mathcal{E}^{\tilde{\alpha}}$ because there the $\mathcal{B}^{2\tilde{\alpha}}$ -norm of $R_{\bar{u}}$ may only blow up at ε^2 but not at every multiple time $kt_* + \varepsilon^2$. For ε small enough this issue can be avoided if we additionally restart the process at times $\frac{2k+1}{2}t_*$ and for every s take the better bound.

In this way we finally obtain the bounds

$$\mathbb{E} \left[\left\| R_{\bar{u}}(s) - R_{u_\varepsilon}(s) \right\|_{\mathcal{B}_{[\varepsilon^2, 2t_*]}^{2\tilde{\alpha}}}^p \left(\mathbf{1}_{\mathfrak{X}_{t_*/2}} \mathbf{1}_{\mathfrak{X}_{t_*}} \mathbf{1}_{\mathfrak{X}_{3t_*/2}} \cdots \mathbf{1}_{\mathfrak{X}_{N_*t_*/2}} \right) \right]^{1/p} \\ \lesssim \varepsilon^\gamma + |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}}.$$

In the same way we get the bounds

$$\mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[\varepsilon^2, t_\varepsilon]}^{\tilde{\alpha}}}^p \left(\mathbf{1}_{\mathfrak{X}_{t_*/2}} \mathbf{1}_{\mathfrak{X}_{t_*}} \mathbf{1}_{\mathfrak{X}_{3t_*/2}} \cdots \mathbf{1}_{\mathfrak{X}_{N_*t_*/2}} \right) \right]^{1/p} \\ \lesssim \varepsilon^\gamma + |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}},$$

and

$$\mathbb{E} \left[\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{t_\varepsilon}^{\tilde{\alpha}}}^p \left(\mathbf{1}_{\mathfrak{X}_{t_*/2}} \mathbf{1}_{\mathfrak{X}_{t_*}} \mathbf{1}_{\mathfrak{X}_{3t_*/2}} \cdots \mathbf{1}_{\mathfrak{X}_{N_*t_*/2}} \right) \right]^{1/p} \\ \lesssim \varepsilon^\gamma + |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}}.$$

Here the events $\mathfrak{X}_{kt_*/2}$ are defined as in (2.38) and $N_* := \lfloor 2T/t_* \rfloor$.

Now using Chebysheff's inequality we get for any $\tilde{\gamma} < \gamma$

$$\mathbb{P} \left[\left(\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[\varepsilon^2, T_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}} + \left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{T_\varepsilon}^{\tilde{\alpha}}} + \left\| R_{\bar{u}} - R_{u_\varepsilon} \right\|_{\mathcal{B}_{[\varepsilon^2, T_\varepsilon]}^{2\tilde{\alpha}}, \tilde{\beta}/2} \geq \varepsilon^{\tilde{\gamma}} \right) \right. \\ \left. \cap \mathfrak{X}_{t_*/2} \cap \dots \cap \mathfrak{X}_{N_*t_*/2} \right] \tag{2.39} \\ \leq \varepsilon^{-\tilde{\gamma}p} \mathbb{E} \left[\left(\left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{[\varepsilon^2, T_\varepsilon]}^{\tilde{\alpha}/2, \tilde{\alpha}}} + \left\| \bar{u} - u_\varepsilon \right\|_{\mathcal{C}_{T_\varepsilon}^{\tilde{\alpha}}} + \left\| R_{\bar{u}} - R_{u_\varepsilon} \right\|_{\mathcal{B}_{[\varepsilon^2, T_\varepsilon]}^{2\tilde{\alpha}}, \tilde{\beta}/2} \right) \right. \\ \left. \left(\mathbf{1}_{\mathfrak{X}_{t_*/2}} \mathbf{1}_{\mathfrak{X}_{t_*}} \mathbf{1}_{\mathfrak{X}_{3t_*/2}} \cdots \mathbf{1}_{\mathfrak{X}_{N_*t_*/2}} \right) \right] \\ \lesssim \varepsilon^{-\tilde{\gamma}p} (\varepsilon^{\gamma p} + |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}}) \rightarrow 0.$$

Thus, in order to conclude the desired estimate (2.23) it only remains to show that the probability of each of the finitely many sets $\mathfrak{X}_{kt_*/2}$ goes to one.

For simplicity we will restrict ourselves to the set $\mathfrak{X}_{t_*/2}$ but the argument for the other sets is the same. In (2.37) we had already seen that

$$\mathbb{E} \left[|\bar{u}(t_*/2) - u_\varepsilon(t_*/2)|_{\mathcal{C}^{\tilde{\alpha}}}^p \right] + \mathbb{E} \left[t_*^{\tilde{\beta}} |R_{\bar{u}}(t_*/2) - R_{u_\varepsilon}(t_*/2)|_{2\tilde{\alpha}}^p \right] \\ \leq K_* \varepsilon^\gamma + K_* |u^0 - u_\varepsilon^0|_{\mathcal{C}^{\tilde{\alpha}}}.$$

In order to conclude that this implies that $\bar{u}(t_*/2)$ is also close to $u_\varepsilon(t_*/2)$ in \mathcal{C}^{α_*} with high probability, we need the following trick: \bar{u} and u_ε are controlled rough paths and we have

$$\delta \bar{u}(t_*/2; x, y) = \theta(\bar{u}(t_*/2, x)) \delta X(t_*/2; x, y) + R_{\bar{u}}(t_*/2; x, y) \\ \delta u_\varepsilon(t_*/2; x, y) = \theta(u_\varepsilon(t_*/2, x)) \delta X_\varepsilon(t_*/2; x, y) + R_{u_\varepsilon}(t_*/2; x, y).$$

Using this decomposition we can conclude that in order to prove convergence in probability in $\mathcal{C}^{\alpha*}$ it suffices to prove convergence of $\theta(u_\varepsilon)$ in \mathcal{C} , X_ε in $\mathcal{C}^{\alpha*}$ as well as R_{u_ε} in $\mathcal{B}^{\alpha*}$. These bounds are readily accounted for. Hence we can conclude the proof of Theorem 2.3. \square

Proof of Theorem 1.5. In order to conclude Theorem 1.5 it is sufficient to show that we have

$$\lim_{K \uparrow \infty} \lim_{\varepsilon \downarrow 0} \mathbb{P} \left[\sup_{0 \leq s \leq \tau_K^*} \|\bar{u} - u_\varepsilon\|_{\mathcal{C}} \geq \varepsilon^{\tilde{\gamma}} \right] = 0. \quad (2.40)$$

Indeed, then the sequence τ_ε can be chosen as a suitable diagonal sequence.

Recall the definitions of the stopping times $\varrho_{K,\varepsilon}$ and σ_K in (2.20) and (2.22).

In order to see 2.40 we write for any $\bar{K} > K$

$$\begin{aligned} \mathbb{P} \left[\sup_{0 \leq s \leq \tau_K^*} \|\bar{u} - u_\varepsilon\|_{\mathcal{C}} \geq \varepsilon^{\tilde{\gamma}} \right] &\leq \mathbb{P} \left[\sup_{0 \leq s \leq \varrho_{\bar{K},\varepsilon}^*} \|\bar{u} - u_\varepsilon\|_{\mathcal{C}} \geq \varepsilon^{\tilde{\gamma}} \right] \\ &\quad + \mathbb{P} \left[\varrho_{\bar{K},\varepsilon} < \sigma_{\bar{K}} \right] + \mathbb{P} \left[\sigma_{\bar{K}} < \tau_K \right]. \end{aligned} \quad (2.41)$$

The first term on the right-hand side has already been bounded in Theorem 2.3. In order to obtain the optimal rate, here we choose $\tilde{\alpha}$ to be as small as possible, i.e. just a bit larger than $\frac{1}{3}$. In particular, by choosing κ small enough we can increase γ up to arbitrarily close to $\frac{1}{6}$.

According to the definition of the stopping times we have

$$\begin{aligned} &\mathbb{P} \left[\varrho_{\bar{K},\varepsilon} < \sigma_{\bar{K}} \right] \\ &= \mathbb{P} \left[\|X - X_\varepsilon\|_{\mathcal{C}_{\varrho_{\bar{K},\varepsilon}}^{\alpha/2,\alpha}} \geq 1, \text{ or } \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}_{\varrho_{\bar{K},\varepsilon}}^{2\alpha}} \geq 1, \right. \\ &\quad \text{or } \sup_{0 \leq t \leq \varrho_{\bar{K},\varepsilon}} \left| D_\varepsilon \mathbf{X}_\varepsilon(t) - \Lambda \text{Id} \right|_{H^{-\alpha}} \geq 1, \text{ or } \|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{[\varepsilon^2, \varrho_{\bar{K},\varepsilon}]}^{\alpha/2,\alpha}} \geq 1, \\ &\quad \text{or } \|\bar{u} - u_\varepsilon\|_{\mathcal{C}_{\varrho_{\bar{K},\varepsilon}}^\alpha} \geq 1, \text{ or } \sup_{\varepsilon^2 < t \leq \varrho_{\bar{K},\varepsilon}} (t - \varepsilon^2)^{\frac{\beta}{2}} |R_{\bar{u}}(t) - R_{u_\varepsilon}(t)|_{2\alpha} \geq 1, \\ &\quad \left. \text{or } \sup_{0 < t \leq \varrho_{\bar{K},\varepsilon}} t^{\frac{\beta}{2}} |R_{\bar{u}}(t) - R_{u_\varepsilon}(t)|_{2\tilde{\alpha}} \geq 1 \right]. \end{aligned} \quad (2.42)$$

We have already provided all the bounds that imply that for any \bar{K} this probability goes to zero. In fact, the bounds for $X - X_\varepsilon$, $\|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}_t^{2\alpha}}$ and $|D_\varepsilon \mathbf{X}_\varepsilon(t) - \Lambda \text{Id}|_{H^{-\alpha}}$ are independent of \bar{K} and given in Corollary 3.3 and Proposition 4.1.

The bounds for the remaining terms in (2.42) follow from applying Theorem 2.3 again – once with $\tilde{\alpha} = \alpha$ and once for arbitrary $\tilde{\alpha}$. Note that it is crucial for this argument, that we allow for the choice $\tilde{\alpha} = \alpha$. In this case the convergence is very slow, but this does not matter for the argument.

Finally, we write for the last term on the right-hand side of (2.41) that

$$\begin{aligned} \mathbb{P} \left[\sigma_{\bar{K}} < \tau_K \right] &= \mathbb{P} \left[\|X\|_{\mathcal{C}_{\tau_K}^{\alpha/2,\alpha}} \geq \bar{K}, \text{ or } \|\mathbf{X}\|_{\mathcal{B}_{\tau_K}^{2\alpha}} \geq \bar{K}, \text{ or } \|\bar{u}\|_{\mathcal{C}_{\tau_K}^{\alpha/2,\alpha}} \geq \bar{K}, \right. \\ &\quad \left. \text{or } \sup_{0 < t \leq \tau_K} t^{\frac{\beta}{2}} |R_{\bar{u}}(t)|_{2\alpha} \geq \bar{K}, \text{ or } \sup_{0 < t \leq \tau_K} t^{\frac{\beta}{2}} |R_{\bar{u}}(t)|_{2\tilde{\alpha}} \geq \bar{K} \right]. \end{aligned}$$

It follows from the bounds in Corollary 3.4 that the probability of $\|X\|_{\mathcal{C}_{\tau_K}^{\alpha/2,\alpha}} \geq \bar{K}$ and the probability of $\|\mathbf{X}\|_{\mathcal{B}_t^{2\alpha}} \geq \bar{K}$ go to zero as \bar{K} goes to ∞ . The same

statement about the probabilities involving \bar{u} and $R_{\bar{u}}$ follows from the global well-posedness of the equation with bounded g, θ . The details of this calculation can be found in the proof of Theorem 3.5 in [HW10] and will be omitted here.

This finishes the proof of our main result, Theorem 1.5. \square

3. THE STOCHASTIC CONVOLUTION

In this section we provide the necessary bounds on the stochastic convolutions $\Psi^{\theta(u)}$ and $\Psi_{\varepsilon}^{\theta(\bar{u})}$. We will adopt a slightly more general framework than the one adopted in Section 2. Actually, we will fix an adapted $L^2[-\pi, \pi]$ -valued process $(\theta(t))_{t \geq 0}$ and consider the stochastic convolutions with the heat semigroup, i.e.

$$\Psi_{\varepsilon}^{\theta}(t) = \int_0^t S_{\varepsilon}(t-s) \theta(s) H_{\varepsilon} dW(s) \quad \text{and} \quad \Psi^{\theta}(t) = \int_0^t S(t-s) \theta(s) dW(s).$$

As in Section 2, the Gaussian case $\theta \equiv 1$ will play a special role and we will denote it by

$$X_{\varepsilon}(t) = \int_0^t S_{\varepsilon}(t-s) H_{\varepsilon} dW(s) \quad \text{and} \quad X(t) = \int_0^t S(t-s) dW(s).$$

It will be useful to consider the Fourier expansion of X_{ε} given by

$$\begin{aligned} X_{\varepsilon}(t, x) &= \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \int_0^t e^{ikx} e^{-k^2 f(\varepsilon k)(t-s)} h(\varepsilon k) dw_k(s) \\ &= \sum_{k \in \mathbb{Z}} q_{\varepsilon}^k \xi_{\varepsilon}^k(t) e^{ikx}. \end{aligned} \tag{3.1}$$

Here the w_k are \mathbb{C}^n -valued standard Brownian motions (i.e. real and imaginary part of every component are independent real-valued Brownian motions so that $\mathbb{E}|w_k^i(t)|^2 = t$), which are independent up to the constraint $w_k = \bar{w}_{-k}$ that ensures that X_{ε} is real-valued. Furthermore we use the notation

$$q_{\varepsilon}^k = \begin{cases} \frac{h(\varepsilon k)}{|k| \sqrt{4\pi f(\varepsilon k)}} & \text{for } k \neq 0 \\ \frac{1}{\sqrt{2\pi}} & \text{for } k = 0 \end{cases} \tag{3.2}$$

and for $k \neq 0$ the ξ_{ε}^k are centred \mathbb{C}^n -valued Gaussian processes, independent up to $\xi_{\varepsilon}^k = \bar{\xi}_{\varepsilon}^{-k}$, so that for $s \leq t$,

$$\mathbb{E}[\xi_{\varepsilon}^k(s) \otimes \xi_{\varepsilon}^{-k}(t)] = \mathcal{K}_k^{s,t} \text{Id}, \tag{3.3}$$

where

$$\mathcal{K}_k^{s,t} = \begin{cases} e^{-f(\varepsilon k)k^2|t-s|} - e^{-f(\varepsilon k)k^2(t+s)}, & k \neq 0, \\ s \wedge t, & k = 0. \end{cases}$$

The series decomposition (3.1) can be used to define the iterated integral

$$\mathbf{X}_{\varepsilon}(t; x, y) = \int_x^y (X_{\varepsilon}(t, z) - X_{\varepsilon}(t, x)) \otimes d_z X_{\varepsilon}(t, z). \tag{3.4}$$

In fact, for fixed t, x, y this integral can simply be defined as the limit in $L^2(\Omega)$ of the double series

$$\begin{aligned} \mathbf{X}_\varepsilon(t; x, y) &= \sum_{k, l \in \mathbb{Z}} \xi_\varepsilon^k(t) \otimes \xi_\varepsilon^l(t) q_\varepsilon^k q_\varepsilon^l \int_x^y (e^{ikz} - e^{ikx}) il e^{ilz} dz \\ &= \sum_{k, l \in \mathbb{Z}} \xi_\varepsilon^k(t) \otimes \xi_\varepsilon^l(t) e^{ikx} q_\varepsilon^k q_\varepsilon^l I_{k, l}(y - x), \end{aligned} \quad (3.5)$$

where

$$I_{k, l}(y) = \begin{cases} \frac{l}{k+l} [e^{i(k+l)y} - 1] - [e^{ily} - 1] & \text{for } k \neq -l \\ ily - [e^{ily} - 1] & \text{for } k = -l. \end{cases} \quad (3.6)$$

The regularity properties of \mathbf{X}_ε are discussed in Lemma 3.3. Note however, that the regularity of X_ε is not sufficient to give a pathwise argument for the existence of (3.4). For the moment let us only point out that the iterated integrals \mathbf{X}_ε satisfy the consistency relation (A.3), and that for the symmetric part $\mathbf{X}_\varepsilon^+ := \frac{1}{2}(\mathbf{X}_\varepsilon + \mathbf{X}_\varepsilon^*)$ we have

$$\mathbf{X}_\varepsilon^+(t; x, y) = \frac{1}{2}(X_\varepsilon(t, y) - X_\varepsilon(t, x)) \otimes (X_\varepsilon(t, y) - X_\varepsilon(t, x)). \quad (3.7)$$

These relations can easily be checked by truncation. The regularity results given in Lemma 3.3 will then imply that for every t the pair $(X_\varepsilon(t, \cdot), \mathbf{X}_\varepsilon(t : \cdot, \cdot))$ is indeed a geometric rough path in x in the sense of definition A.1.

A crucial tool to derive the moment estimates for \mathbf{X}_ε will be the equivalence of moments for random variables in a given Wiener chaos. The decomposition (3.5) shows that \mathbf{X}_ε is in the second Wiener chaos. Therefore, the Nelson estimate implies that we can estimate all moments of \mathbf{X}_ε in terms of the second moments.

Note that our definition of \mathbf{X}_ε coincides with the canonical rough path lift of a Gaussian process discussed in [FV10, Ch. 15] and also used in [Hai11a, HW10]. We prefer to work with the Fourier decomposition as it gives a direct way to prove moment bounds and avoids the notion of 2-dimensional variation of the covariance, which seems a bit cumbersome in the present context.

A key step in the construction of solutions to (1.1) in [HW10] was to show that the process $\Psi^\theta(t, \cdot)$ is controlled by $X(t, \cdot)$ as soon as θ has a certain regularity. The derivative process is given as $\theta(t, \cdot)$.

We will prove a similar statement for Ψ_ε^θ and derive bounds that are uniform in ε . For a given θ denote by R_ε^θ the remainder in the rough path decomposition of Ψ_ε^θ with respect to X_ε , i.e.

$$R_\varepsilon^\theta(t, x, y) = (\Psi_\varepsilon^\theta(t, y) - \Psi_\varepsilon^\theta(t, x)) - \theta(t, x)(X_\varepsilon(t, y) - X_\varepsilon(t, x)) \quad (3.8)$$

and

$$R^\theta(t, x, y) = (\Psi^\theta(t, y) - \Psi^\theta(t, x)) - \theta(t, x)(X(t, y) - X(t, x)). \quad (3.9)$$

From the definition of Ψ_ε^θ and X_ε we get

$$R_\varepsilon^\theta(t, x, y) = \int_0^t \int_{-\pi}^\pi (p_{t-s}^\varepsilon(y-z) - p_{t-s}^\varepsilon(x-z)) (\theta(s, z) - \theta(t, x)) H_\varepsilon dW(s, z) \quad (3.10)$$

and similarly for R^θ . The bounds on the space time regularity of R_ε^θ are provided in Lemma 3.6. A key tool to derive these a priori bounds is provided by a higher-dimensional version of the Garsia-Rodemich-Rumsey Lemma that can be found in Lemma B.3.

For the bound on Ψ_ε^θ we will impose a regularity assumption on θ . For any stopping time τ recall the definition of the parabolic α -Hölder norm in (1.15).

Lemma 3.1. *Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Let $\alpha_1, \alpha_2 > 0$ and $p \geq 2$ satisfy*

$$\alpha_1 < \frac{\lambda_1}{4} - \frac{1}{p}, \quad \alpha_2 < \frac{\lambda_2}{2} - \frac{1}{p} \quad (3.11)$$

for some $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$. Then for any stopping time $\tau \leq T$

$$\mathbb{E} \|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\alpha_1}(\mathcal{C}^{\alpha_2})}^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p. \quad (3.12)$$

and

$$\mathbb{E} \|\Psi_\varepsilon^\theta - \Psi^\theta\|_{\mathcal{C}_\tau^{\alpha_1}(\mathcal{C}^{\alpha_2})}^p \lesssim \varepsilon^{(1-\lambda_1-\lambda_2)\alpha p} \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p. \quad (3.13)$$

In our application of this lemma we shall need a small power of T appearing in the right-hand side. This additional factor can be easily obtained by observing that as $\Psi_\varepsilon^\theta(0) = 0$ we have

$$\|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\alpha_1-\kappa}(\mathcal{C}^{\alpha_2})} \leq T^\kappa \|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\alpha_1}(\mathcal{C}^{\alpha_2})}.$$

Furthermore, we prefer to work with the space-time Hölder norms introduced in (1.15), instead of working in spaces of functions that are Hölder in time taking values in a Hölder space. To this end we observe that

$$\|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\alpha/2, \alpha}} \leq \|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\alpha/2}(\mathcal{C})} + \|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau(\mathcal{C}^\alpha)}.$$

In view of these remarks, the following result is an easy consequence of Lemma 3.1.

Corollary 3.2. *Let $\alpha, \tilde{\alpha} \in (\frac{1}{3}, \frac{1}{2})$. Suppose that p satisfies*

$$p > \frac{6}{1-2\tilde{\alpha}}.$$

Then for $\lambda = 1 - 2\tilde{\alpha} - \frac{6}{p}$ and for any stopping time $\tau \leq T$ we have

$$\mathbb{E} \|\Psi_\varepsilon^\theta\|_{\mathcal{C}_\tau^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \lesssim T^{\lambda p/4} \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p \quad (3.14)$$

and

$$\mathbb{E} \|\Psi_\varepsilon^\theta - \Psi^\theta\|_{\mathcal{C}_\tau^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \lesssim \varepsilon^{\lambda p \alpha} \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p. \quad (3.15)$$

Proof of Lemma 3.1. Lemma B.3 applied to $F = \Psi_\varepsilon^\theta - \Psi^\theta$ will imply the desired bound (3.12) as soon as we have established the following inequalities

$$\mathbb{E} |\Psi_\varepsilon^\theta(t_1 \wedge \tau, x) - \Psi_\varepsilon^\theta(t_2 \wedge \tau, x)|^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p |t_1 - t_2|^{\frac{p}{4}}, \quad (3.16a)$$

$$\mathbb{E} |\Psi_\varepsilon^\theta(t \wedge \tau, x_1) - \Psi_\varepsilon^\theta(t \wedge \tau, x_2)|^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p |x_1 - x_2|^{\frac{p}{2}}, \quad (3.16b)$$

$$\mathbb{E} |\Psi_\varepsilon^\theta(t \wedge \tau, x)|^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p. \quad (3.16c)$$

Then (3.13) will follow as soon as we establish in addition that

$$\mathbb{E} |\Psi_\varepsilon^\theta(t \wedge \tau, x) - \Psi^\theta(t \wedge \tau, x)|^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p \varepsilon^{\alpha' p} \quad (3.16d)$$

for any $\alpha' < \alpha$. We state (3.16a) and (3.16b) only for Ψ_ε^θ noting that Ψ^θ is included as the special case $\varepsilon = 0$. In the calculations we will use the abbreviated notation $t^\tau = t \wedge \tau$. To see (3.16a) we can write for $t_1 \geq t_2$,

$$\begin{aligned} & \Psi_\varepsilon^\theta(t_1^\tau, x) - \Psi_\varepsilon^\theta(t_2^\tau, x) \\ &= \int_0^{t_2^\tau} \int_{-\pi}^\pi H_\varepsilon \left((p_{t_1^\tau-s}^\varepsilon(x - \cdot) - p_{t_2^\tau-s}^\varepsilon(x - \cdot)) \theta(s, \cdot) \right) dW(s) \\ &+ \int_{t_2^\tau}^{t_1^\tau} \int_{-\pi}^\pi H_\varepsilon \left(p_{t_1^\tau-s}^\varepsilon(x - \cdot) \theta(s, \cdot) \right) dW(s). \end{aligned}$$

Here we recall the definitions of the heat kernel p_t^ε in (2.9) and the smoothing operator H_ε in Assumption 1.4.

Using the Burkholder-Davis-Gundy inequality ([KS91, Theorem 3.28]) we get

$$\begin{aligned} & \mathbb{E} |\Psi_\varepsilon^\theta(t_1^\tau, x) - \Psi_\varepsilon^\theta(t_2^\tau, x)|^p \tag{3.17} \\ & \lesssim \mathbb{E} \left(\int_0^{t_2^\tau} \int_{-\pi}^\pi \left[H_\varepsilon \left((p_{t_1^\tau-s}^\varepsilon(x - \cdot) - p_{t_2^\tau-s}^\varepsilon(x - \cdot)) \theta(s, \cdot) \right) (y) \right]^2 ds dy \right)^{\frac{p}{2}} \\ &+ \mathbb{E} \left(\int_{t_2^\tau}^{t_1^\tau} \int_{-\pi}^\pi \left[H_\varepsilon \left(p_{t_1^\tau-s}^\varepsilon(x - \cdot) \theta(s, \cdot) \right) (y) \right]^2 ds dy \right)^{\frac{p}{2}}. \end{aligned}$$

Observing that due to the boundedness of h convolution with H_ε is uniformly bounded as an operator on L^2 we can bound the first expectation on the right-hand side of (3.17) by a constant times

$$\mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p \left(\int_0^{t_2} \int_{-\pi}^\pi (p_{t_1-s}^\varepsilon(x - y) - p_{t_2-s}^\varepsilon(x - y))^2 ds dy \right)^{\frac{p}{2}}. \tag{3.18}$$

Using Parseval's identity the double integral in (3.18) can be bounded by

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \left(e^{-k^2 f(\varepsilon k) (t_1 - t_2)} - 1 \right)^2 \int_0^{t_2} e^{-2k^2 f(\varepsilon k) (t_2 - s)} ds \\ & \lesssim \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(k^2 f(\varepsilon k) (t_1 - t_2) \wedge 1 \right)^2 \frac{1}{k^2 f(\varepsilon k)} \\ & \lesssim \sum_{k \in \mathbb{Z} \setminus \{0\}} (t_1 - t_2) \wedge \frac{1}{k^2 f(\varepsilon k)} \\ & \lesssim \sum_{|k| \leq (t_1 - t_2)^{-1/2}} (t_1 - t_2) + \sum_{|k| > (t_1 - t_2)^{-1/2}} \frac{1}{k^2} \lesssim (t_1 - t_2)^{\frac{1}{2}}. \end{aligned}$$

Here in the second inequality we have used that $|a|^2 \leq |a|$ whenever $|a| \leq 1$, and in the third inequality we used the assumption that f is bounded from below.

The second integral on the right-hand side of (3.17) can be bounded in a similar way by

$$\begin{aligned} & \mathbb{E} \left(\int_{t_2^\tau}^{t_1^\tau} \int_{-\pi}^\pi \left(H_\varepsilon (p_{t_1^\tau-s}^\varepsilon(x - \cdot) \theta(s, \cdot)) (y) \right)^2 ds dy \right)^{\frac{p}{2}} \\ & \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p \left(\int_{t_2}^{t_1} \int_{-\pi}^\pi p_{t_1-s}^\varepsilon(x - y)^2 ds dz \right)^{\frac{p}{2}}. \end{aligned}$$

To bound the integral we calculate

$$\begin{aligned}
\int_{t_2}^{t_1} \int_{-\pi}^{\pi} p_{t_1-s}^{\varepsilon}(x-y)^2 ds dy &= \sum_{k \in \mathbb{Z}} \int_{t_2}^{t_1} e^{-2k^2 f(\varepsilon k)(t_1-s)} ds \\
&\lesssim \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{k^2 f(\varepsilon k)(t_1 - t_2) \wedge 1}{k^2 f(\varepsilon k)} + (t_1 - t_2) \\
&\lesssim (t_1 - t_2)^{\frac{1}{2}}.
\end{aligned}$$

This finishes the proof of (3.16a).

Using the Burkholder-Davis-Gundy inequality and the uniform L^2 -boundedness of the convolution with H_{ε} in the same way as before, the derivation of (3.16b) can be reduced to showing that

$$\int_0^t \int_{-\pi}^{\pi} (p_{t-s}^{\varepsilon}(x_1 - y) - p_{t-s}^{\varepsilon}(x_2 - y))^2 dy ds \lesssim |x_1 - x_2|.$$

To prove the latter bound, we estimate

$$\begin{aligned}
&\int_0^t \int_{-\pi}^{\pi} (p_{t-s}^{\varepsilon}(x_1 - z) - p_{t-s}^{\varepsilon}(x_2 - z))^2 dy ds \\
&= \sum_{k \in \mathbb{Z} \setminus \{0\}} |e^{ik(x_2 - x_1)} - 1|^2 \int_0^t e^{-k^2 f(\varepsilon k)(t-s)} ds \\
&\lesssim \sum_{k \in \mathbb{Z} \setminus \{0\}} (k|x_1 - x_2| \wedge 1)^2 \frac{1}{k^2 f(\varepsilon k)} \\
&\lesssim \sum_{k \leq |x_1 - x_2|^{-1}} |x_1 - x_2|^2 + \sum_{k > |x_1 - x_2|^{-1}} \frac{1}{k^2} \lesssim |x_1 - x_2|.
\end{aligned}$$

This shows (3.16b).

The bound (3.16c) follows immediatly, by using the Burkholder-Davis-Gundy inequality in the same way as above.

In order to obtain (3.16d) we write

$$\begin{aligned}
&\Psi_{\varepsilon}^{\theta}(t^{\tau}, x) - \Psi^{\theta}(t^{\tau}, x) \\
&= \int_0^{t^{\tau}} \int_{-\pi}^{\pi} H_{\varepsilon} \left((p_{t^{\tau}-s}^{\varepsilon}(x - \cdot) - p_{t^{\tau}-s}(x - \cdot)) \theta(s, \cdot) \right) (y) dW(s, y) \\
&\quad + \int_0^{t^{\tau}} \int_{-\pi}^{\pi} (\text{Id} - H_{\varepsilon}) (p_{t^{\tau}-s}(x, \cdot) \theta(s, \cdot)) (y) dW(s, y).
\end{aligned} \tag{3.19}$$

The first term on the right-hand side of (3.19) can be treated as before. Up to a constant its p -th moment is bounded by

$$\mathbb{E} \|\theta\|_{C^{\tau}}^p \left(\sum_{k \in \mathbb{Z}} \int_0^t \left(e^{-(t-s)k^2 f(\varepsilon k)} - e^{-(t-s)k^2} \right)^2 ds \right)^{\frac{p}{2}}. \tag{3.20}$$

To get a bound on (3.20) we write

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}} \int_0^t \left(e^{-(t-s)k^2 f(\varepsilon k)} - e^{-(t-s)k^2} \right)^2 ds \\
&= \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_0^t e^{-2k^2 c_f s} \left(e^{-k^2(f(\varepsilon k) - c_f)s} - e^{-k^2(1 - c_f)s} \right)^2 ds \\
&\leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_0^t e^{-2k^2 c_f s} (1 \wedge s k^2 |f(\varepsilon k) - 1|)^2 ds.
\end{aligned} \tag{3.21}$$

Recall that the constant c_f is defined in Assumption 1.1. Using Assumption 1.1 on f once more, one can see that for $|\varepsilon k| < \delta$ we have

$$1 \wedge s k^2 |f(\varepsilon k) - 1| \lesssim 1 \wedge |s k^2 \varepsilon k|.$$

Hence, up to a constant the sum in (3.21) can then be bounded by

$$\sum_{0 < |k| \leq \delta \varepsilon^{-1}} \int_0^t e^{-2k^2 c_f s} s^2 k^4 (\varepsilon^2 k^2) ds + \sum_{\delta \varepsilon^{-1} < |k|} \int_0^t e^{-2k^2 c_f s} ds \lesssim \varepsilon.$$

Finally, to treat the second term in (3.19) we need to impose a stronger regularity assumption on θ . We have the identity

$$|\theta(s, \cdot) p_{t-s}(x - \cdot)|_{H^{\alpha'}} \lesssim |\theta(s, \cdot)|_{C^\alpha} |p_{t-s}(x - \cdot)|_{H^\alpha} \tag{3.22}$$

which holds for every $\alpha' < \alpha$. In fact, to see (3.22) write

$$\begin{aligned}
& \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|\theta p_{t-s}(x_1) - \theta p_{t-s}(x_2)|^2}{|x_1 - x_2|^{2\alpha'+1}} dx_1 dx_2 \\
&\lesssim \sup_x |\theta(x)|^2 |p_{t-s}|_{H^{\alpha'}}^2 + |\theta|_{C^\alpha}^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|x_1 - x_2|^{2\alpha} |p_{t-s}(x_2)|^2}{|x_1 - x_2|^{2\alpha'+1}} dx_1 dx_2 \\
&\lesssim \sup_x |\theta(x)|^2 |p_{t-s}|_{H^{\alpha'}}^2 + |\theta|_{C^\alpha}^2 |p_{t-s}|_{L^2}^2.
\end{aligned}$$

Then the Burkholder-Davis-Gundy inequality yields

$$\begin{aligned}
& \mathbb{E} \left| \int_0^{t^\tau} \int_{-\pi}^{\pi} \left((\text{Id} - H^\varepsilon) p_{t^\tau-s}(x - \cdot) \theta(s, \cdot) \right) (y) dW(s, y) \right|^p \\
&\lesssim \mathbb{E} \left[\left(\sup_{s \in [0, T]} |\theta(s, \cdot)|_{C^\alpha} \right)^p \left(\int_0^t |p_{t-s}|_{H^\alpha}^2 |\text{Id} - H^\varepsilon|_{H^{\alpha'} \rightarrow L^2}^2 ds \right)^{\frac{p}{2}} \right].
\end{aligned} \tag{3.23}$$

Now for any $\alpha \in (0, 1)$

$$|p_t(x - \cdot)|_{H^\alpha}^2 \approx \sum_{k \in \mathbb{Z}} (1 + |k|^{2\alpha}) e^{-2tk^2} \lesssim t^{-\frac{1}{2}-\alpha}.$$

On the other hand

$$|\text{Id} - H^\varepsilon|_{H^{\alpha'} \rightarrow L^2} \approx \sup_{k \in \mathbb{Z}} \frac{1 - h(\varepsilon|k|)}{1 + |k|^{\alpha'}} \leq \varepsilon^{\alpha'} \sup_{r \in \mathbb{R}} \frac{1 - h(|r|)}{\varepsilon^{\alpha'} + |r|^{\alpha'}} \lesssim \varepsilon^{\alpha'},$$

due to the assumptions on h 1.4. Due to the assumption $\alpha < \frac{1}{2}$ the right-hand side of (3.23) is thus integrable and we arrive at (3.16d). This finishes the proof. \square

As a next step we give bounds on the approximation of the Gaussian rough path (X, \mathbf{X}) .

Lemma 3.3. *For any $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, any $\varepsilon \geq 0$ and any t , the pair $(X_\varepsilon(t, \cdot), \mathbf{X}_\varepsilon(t; \cdot))$ is a geometric α -rough path in the sense of Definition A.1.*

Furthermore, let $p \in [1, \infty)$ and let $\alpha_1, \alpha_2 > 0$ satisfy

$$\alpha_1 < \frac{\lambda_1}{4}, \quad \alpha_2 < \frac{\lambda_2}{2} \quad (3.24)$$

for some $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$. Then we have for any $\varepsilon \geq 0$

$$\mathbb{E} \|X_\varepsilon\|_{C_T^{\alpha_1}(C^{\alpha_2})}^p \lesssim 1, \quad (3.25)$$

$$\mathbb{E} \|X_\varepsilon - X\|_{C_T^{\alpha_1}(C^{\alpha_2})}^p \lesssim \varepsilon^{\frac{1}{2}(1-\lambda_1-\lambda_2)p}, \quad (3.26)$$

and

$$\mathbb{E} \|\mathbf{X}_\varepsilon\|_{C_T^{\alpha_1}(B^{2\alpha_2})}^p \lesssim 1, \quad (3.27)$$

$$\mathbb{E} \|\mathbf{X}_\varepsilon - \mathbf{X}\|_{C_T^{\alpha_1}(B^{2\alpha_2})}^p \lesssim \varepsilon^{\frac{1}{2}(1-\lambda_1-\lambda_2)p}. \quad (3.28)$$

We will need uniform in time estimates on \mathbf{X}_ε and we will not make use of the Hölder in time regularity provided by Lemma 3.3. Therefore, we will actually use the following version of Lemma 3.3.

Corollary 3.4. *Let $\tilde{\alpha} \in (\frac{1}{3}, \frac{1}{2})$. Suppose that $\lambda < 1 - 2\tilde{\alpha}$. Then for any $p \geq 1$ and $T > 0$ we have*

$$\mathbb{E} \|X_\varepsilon\|_{C_T^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \lesssim T^{\frac{\lambda p}{4}}, \quad (3.29)$$

$$\mathbb{E} \|X_\varepsilon - X\|_{C_T^{\tilde{\alpha}/2, \tilde{\alpha}}}^p \lesssim \varepsilon^{\frac{\lambda p}{2}}. \quad (3.30)$$

Similarly, if and $\tilde{\beta} \in (\frac{1}{2}, 1)$ and $\lambda < \frac{1-\tilde{\beta}}{2}$ we get

$$\mathbb{E} \|\mathbf{X}_\varepsilon\|_{B_T^{\tilde{\beta}}}^p \lesssim T^{\frac{\lambda p}{4}}, \quad (3.31)$$

$$\mathbb{E} \|\mathbf{X}_\varepsilon - \mathbf{X}\|_{B_T^{\tilde{\beta}}}^p \lesssim \varepsilon^{\frac{\lambda p}{2}}. \quad (3.32)$$

Proof of Lemma 3.3. By the monotonicity of L^p -norms, we may assume without loss of generality that (3.24) is replaced by (3.11). The bounds (3.25) and (3.26) on X_ε are then included in Lemma 3.1 as the special case of $\theta = 1$. The consistency relation (A.3) and the symmetry condition were already observed above (see (3.7) and above). Thus it only remains to show (3.27) and (3.28).

To apply Lemma B.3 to the \mathbf{X}_ε we need to prove the following bounds: For every $\gamma < 1$ we have

$$\mathbb{E} |\mathbf{X}_\varepsilon(t; x, y) - \mathbf{X}_\varepsilon(s; x, y)|^p \lesssim |t - s|^{\frac{\gamma p}{4}}, \quad (3.33)$$

$$\mathbb{E} |\mathbf{X}_\varepsilon(t; x, y)|^p \lesssim |x - y|^{\gamma p}, \quad (3.34)$$

$$\mathbb{E} |\delta \mathbf{X}_\varepsilon(t)|_{[x, y]}^p \lesssim |x - y|^{\gamma p}, \quad (3.35)$$

$$\mathbb{E} |\mathbf{X}_\varepsilon(t; x, y) - \mathbf{X}(t; x, y)|^p \lesssim \varepsilon^{\frac{\gamma p}{2}}. \quad (3.36)$$

The bound (3.35) follows directly from the consistency relation (A.3) as for all $x \leq y$

$$\mathbb{E} |\delta \mathbf{X}_\varepsilon(t)|_{[x, y]}^p = \mathbb{E} |\delta X_\varepsilon(t) \otimes \delta X_\varepsilon(t)|_{[x, y]}^p \lesssim |x - y|^\gamma \mathbb{E} |X_\varepsilon(t)|_{\frac{\gamma}{2}}^{2p}. \quad (3.37)$$

Due to Lemma 3.1 we know that the expectation on the right-hand side of (3.37) is finite, which implies (3.35).

For the remaining bounds we will divide $\mathbf{X}_\varepsilon = \mathbf{X}_\varepsilon^+ + \mathbf{X}_\varepsilon^-$ into its symmetric and antisymmetric parts and prove the bounds for the $\mathbf{X}_\varepsilon^\pm$ separately.

For \mathbf{X}_ε^+ we get using (3.7) that

$$\mathbf{X}^+(t; x, y) = \frac{1}{2} (X_\varepsilon(t, y) - X_\varepsilon(t, x)) \otimes (X_\varepsilon(t, y) - X_\varepsilon(t, x))$$

and so (3.33), (3.34) and (3.36) with \mathbf{X}_ε replaced by \mathbf{X}_ε^+ follow directly from (3.16a) – (3.16d).

Let us treat the antisymmetric part \mathbf{X}^- . Due to the equivalence of all moments in the second Wiener chaos (see e.g. [FV10, Thm D.8] or [Nua06]) it is sufficient to prove the bounds in the special case $p = 2$. To derive (3.34) we write

$$\mathbf{X}^-(t; x, y) = \frac{1}{2} \sum_{k \neq -l \in \mathbb{Z}^*} \xi_\varepsilon^k(t) \otimes \xi_\varepsilon^l(t) e^{ikx} q_\varepsilon^k q_\varepsilon^l (I_{k,l}(y-x) - I_{-l,-k}(y-x)).$$

Recall the definitions (3.2) for the q_ε^k , (3.3) for the $\xi_\varepsilon^k(t)$ and (3.6) for the $I_{k,l}$. Then we can write

$$\begin{aligned} \mathbb{E} |\mathbf{X}^-(t; x, y)|^2 &= \frac{1}{4} \sum_{\substack{k \neq -l \in \mathbb{Z}^* \\ \bar{k} \neq -\bar{l} \in \mathbb{Z}^*}} \mathbb{E} [\text{tr}(\xi_\varepsilon^k \otimes \xi_\varepsilon^l)(\xi_\varepsilon^{-\bar{l}} \otimes \xi_\varepsilon^{-\bar{k}})] q_\varepsilon^k q_\varepsilon^l q_\varepsilon^{\bar{k}} q_\varepsilon^{\bar{l}} \\ &\quad \times (I_{k,l}(y-x) - I_{-l,-k}(y-x))(I_{-\bar{k},-\bar{l}}(y-x) - I_{\bar{l},\bar{k}}(y-x)). \end{aligned}$$

For $k \neq -l$ and $\bar{k} \neq -\bar{l}$ we get

$$\mathbb{E} [\text{tr}(\xi_\varepsilon^k \otimes \xi_\varepsilon^l)(\xi_\varepsilon^{-\bar{l}} \otimes \xi_\varepsilon^{-\bar{k}})] = \left(n^2 \delta_{k,\bar{k}} \delta_{l,\bar{l}} + n \delta_{k,\bar{l}} \delta_{l,\bar{k}} \right) \left[1 - e^{-2k^2 f(\varepsilon k)s} \right].$$

Furthermore, we have

$$I_{k,l}(y-x) - I_{-l,-k}(y-x) = 2i \left(\frac{l}{l+k} \sin((k+l)(y-x)) - \sin(l(y-x)) \right).$$

The absolute value of the last expression can be bounded by

$$\begin{aligned} |I_{k,l}(y-x) - I_{-l,-k}(y-x)| &\lesssim \left(\left| \frac{l}{k+l} \right| + 1 \right) \wedge |kl| |y-x|^2 \\ &\lesssim \left(\left| \frac{l}{k+l} \right| + 1 \right)^{1-\tilde{\gamma}} (|kl| |y-x|^2)^{\tilde{\gamma}}, \end{aligned}$$

for every $\tilde{\gamma} \in [0, 1]$. Then we get

$$\mathbb{E} |\mathbf{X}^-(t; x, y)|^2 \lesssim |x-y|^{4\tilde{\gamma}} \sum_{k \neq -l \in \mathbb{Z}^*} \frac{1}{|kl|^{2-2\tilde{\gamma}}} \left(\left| \frac{l}{k+l} \right| + 1 \right)^{2-2\tilde{\gamma}}.$$

The sums appearing in the last line are finite if $\tilde{\gamma} < \frac{1}{4}$. This implies that (3.34) holds also for the antisymmetric part of \mathbf{X}_ε .

To derive the bound (3.33) on the time regularity of \mathbf{X}_ε we write for $t \geq s$

$$\mathbf{X}^-(t) - \mathbf{X}^-(s) = \sum_{k \neq -l \in \mathbb{Z}^*} \left[\xi_\varepsilon^k(t) \otimes \xi_\varepsilon^l(t) - \xi_\varepsilon^k(s) \otimes \xi_\varepsilon^l(s) \right] q_\varepsilon^k q_\varepsilon^l (I_{k,l} - I_{-k,-l}).$$

Using (3.3) we get that for $k \neq -l$, $\bar{k} \neq -\bar{l}$

$$\left| \mathbb{E} \left[\text{tr}(\xi_\varepsilon^k(t) \otimes \xi_\varepsilon^l(t) - \xi_\varepsilon^k(s) \otimes \xi_\varepsilon^l(s)) (\xi_\varepsilon^{-\bar{l}}(t) \otimes \xi_\varepsilon^{-\bar{k}}(t) - \xi_\varepsilon^{-\bar{l}}(s) \otimes \xi_\varepsilon^{-\bar{k}}(s)) \right] \right|$$

$$\lesssim (\delta_{k,-\bar{k}} \delta_{l,-\bar{l}} + \delta_{k,-\bar{l}} \delta_{l,-\bar{k}}) \left[(1 \wedge k^2 f(\varepsilon k) |t-s|) + (1 \wedge l^2 f(\varepsilon l) |t-s|) \right].$$

Therefore, similarly to the calculations for the space regularity we get

$$\begin{aligned} \mathbb{E} |\mathbf{X}_\varepsilon(-t; x, y) - \mathbf{X}_\varepsilon(s; x, y)|^2 & \quad (3.38) \\ & \lesssim \sum_{k \neq -l \in \mathbb{Z}^*} \frac{(1 \wedge f(\varepsilon k) k^2 |t-s|) + (1 \wedge f(\varepsilon l) l^2 |t-s|)}{f(\varepsilon k) k^2 f(\varepsilon l) l^2} \left(\left| \frac{l}{k+l} \right| + 1 \right)^2. \end{aligned}$$

Expanding the products on the right-hand side of (3.38) the most interesting term is

$$\sum_{k \neq -l \in \mathbb{Z}^*} \frac{1 \wedge (f(\varepsilon l) l^2 |t-s|)}{f(\varepsilon k) k^2 f(\varepsilon l) l^2} \left| \frac{l}{k+l} \right|^2 \lesssim |t-s|^{\tilde{\gamma}} \sum_{k \neq -l \in \mathbb{Z}^*} \frac{l^{2\tilde{\gamma}}}{k^2 |l+k|^2}. \quad (3.39)$$

The sum on the right-hand side of (3.39) converges for any $\tilde{\gamma} < \frac{1}{2}$. The other terms on the right-hand side of (3.38) can be bounded up to a constant by $|t-s|^{\frac{1}{2}}$. So (3.33) follows.

Finally, to derive (3.36) using the formula (3.1), we get similarly to (3.37)

$$\begin{aligned} \mathbb{E} |\mathbf{X}_\varepsilon^-(t; x, y) - \mathbf{X}^-(t; x, y)|^2 & \lesssim \sum_{\substack{k \neq -l \in \mathbb{Z}^* \\ \bar{k} \neq -\bar{l} \in \mathbb{Z}^*}} (\delta_{k,-\bar{k}} \delta_{l,-\bar{l}} + \delta_{k,-\bar{l}} \delta_{l,-\bar{k}}) \quad (3.40) \\ & \times \left(\frac{1}{l^2} \mathcal{D}_k^\varepsilon + \frac{1}{k^2} \mathcal{D}_l^\varepsilon \right) |I_{k,l} - I_{-l,-k}| |I_{\bar{k},\bar{l}} - I_{-\bar{l},-\bar{k}}|, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_k^\varepsilon &:= \int_0^t |e^{-k^2 f(\varepsilon |k|)(t-s)} h(\varepsilon k) - e^{-k^2(t-s)}| ds \\ &\lesssim \int_0^t h(\varepsilon k) e^{-k^2 c_f(t-s)} |1 \wedge k^2(t-s)(f(\varepsilon k) - 1)| ds \\ &\quad + \int_0^t e^{-k^2(t-s)} |h(\varepsilon k) - 1| ds \\ &\lesssim \frac{1}{k^2} (1 \wedge \varepsilon^2 k^2). \end{aligned}$$

Then the bound (3.36) can be established as before. \square

Finally, we give the necessary bounds on the remainder term R_ε^θ . The derivation of the uniform bounds is more difficult than in the cases of Ψ_ε^θ or $(X_\varepsilon, \mathbf{X}_\varepsilon)$. Using a space-time regularity assumption on the process θ and the representation (3.10), in [HW10, Lemma 4.2] the regularity of the remainder term in the case $\varepsilon = 0$ is derived from the inequalities

$$\int_0^t \int_{-\pi}^\pi (p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y))^2 |x_1 - y|^{2\alpha} dy ds \lesssim |x_1 - x_2|^{1+2\alpha} \quad (3.41)$$

and

$$\int_0^t \int_{-\pi}^\pi (p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y))^2 |t-s|^\alpha dy ds \lesssim |x_1 - x_2|^{1+2\alpha}. \quad (3.42)$$

To show these identities the heat kernel p_t is rewritten as a linear combination of Gaussian heat kernels using the reflection principle. Then the Gaussian integrals are evaluated explicitly.

In our context two problems arise. First of all the reflection principle is not available for the approximated heat kernel p_t^ε . Parseval's identity that was used above is not well suited to calculate the spatial integral against $|x_1 - y|^{2\alpha}$.

We derive similar identities to (3.41) and (3.42) in Lemma 3.5. The main step is to adapt a statement from [Ste57] to the current context. These calculations can be found in Section 6.

The second issue concerns the lack of time regularity of the process u near zero. Recall that in the application we have in mind we have $\theta = \theta(u_\varepsilon)$. We had seen in Subsections 2.2 and 2.3 that in general the process u_ε need not be time continuous near 0. We only have the necessary “almost $\frac{1}{4}$ ” Hölder regularity for times $t \geq \varepsilon^2$. Hence, we divide the term R_ε^θ into a part for which we can use the regularity of θ and another part in which we use the regularisation of the heat semigroup to obtain the desired regularity.

Lemma 3.5. *The following identities hold*

$$\int_0^t \int_{-\pi}^\pi (p_{t-s}^\varepsilon(x_1 - y) - p_{t-s}^\varepsilon(x_2 - y))^2 |x_1 - y|^{2\alpha} dy ds \lesssim |x_1 - x_2|^{1+2\alpha} \quad (3.43)$$

and

$$\int_0^t \int_{-\pi}^\pi (p_{t-s}^\varepsilon(x_1 - y) - p_{t-s}^\varepsilon(x_2 - y))^2 |t - s|^{2\alpha} dy ds \lesssim |x_1 - x_2|^{1+2\alpha}. \quad (3.44)$$

Before we proceed to the proof of this result, we introduce the auxiliary functions

$$a_{\varepsilon,t}(k) = \exp(-k^2(f(\varepsilon k) - c_f)t), \quad (3.45)$$

together with the corresponding Fourier multipliers $A_\varepsilon(t)$. In the special case $\varepsilon = 0$, we will simply write $A(t) := S((1 - c_f)t)$. The reason for this notation is that we will frequently make use of the factorisation

$$S_\varepsilon(t) = A_\varepsilon(t) S(c_f t).$$

As a simple corollary of Assumption 1.2, it then follows that

$$\sup_{\varepsilon, t \geq 0} \|a_{\varepsilon,t}\|_{\text{BV}} < \infty. \quad (3.46)$$

This is because $a_{\varepsilon,t}(k) = b_{t/\varepsilon^2}(\varepsilon k)$, and the BV-norm is invariant under reparametrisations. With these notations at hand, we now proceed to the proof of Lemma 3.5.

Proof of Lemma 3.5. In order to obtain (3.43) we write for fixed s

$$\begin{aligned} & \int_{-\pi}^\pi (p_{t-s}^\varepsilon(x_1 - y) - p_{t-s}^\varepsilon(x_2 - y))^2 |x_1 - y|^{2\alpha} dy \\ &= \int_{-\pi}^\pi \left(A_\varepsilon(t-s) [p_{c_f(t-s)}(x_1 - \cdot) - p_{c_f(t-s)}(x_2 - \cdot)](y) \right)^2 |x_1 - y|^{2\alpha} dy. \end{aligned}$$

Combining (3.46) with Lemma 6.2, we see that

$$A_\varepsilon(t-s)(x) \lesssim \frac{1}{|x|}.$$

Here, we identify the operator $A_\varepsilon(t-s)$ with its convolution kernel. Since we have $a_{\varepsilon,t-s}(k) \leq 1$ the operator $A_\varepsilon(t-s)$ is a contraction on $L^2[-\pi, \pi]$. So, Lemma 6.1 implies that

$$\begin{aligned} & \int_{-\pi}^{\pi} \left(A_\varepsilon(t-s) [p_{c_f(t-s)}(x_1 - \cdot) - p_{c_f(t-s)}(x_2 - \cdot)](y) \right)^2 |x_1 - y|^{2\alpha} dy \\ & \lesssim \int_{-\pi}^{\pi} (p_{c_f(t-s)}(x_1 - y) - p_{c_f(t-s)}(x_2 - y))^2 |x_1 - y|^{2\alpha} dy. \end{aligned}$$

Note that the constant which is hidden in the \lesssim does not depend on t . Then (3.43) follows from (3.41). In fact, introducing an extra viscosity c_f only changes the constant in (3.41). Using again that $A_\varepsilon(t-s)$ is a contraction on $L^2[-\pi, \pi]$ the bound (3.44) follows directly from (3.42). \square

Now we are able to prove bounds on the R_ε^θ . In order to deal with the lack of temporal regularity of θ for small times we divide R_ε^θ into two parts. We write for $\varepsilon \geq 0$ and for any $s \in [0, T]$

$$R_\varepsilon^\theta(t; x, y) = R_{s^+, \varepsilon}^\theta(t; x, y) + R_{s^-, \varepsilon}^\theta(t; x, y).$$

Here

$$\begin{aligned} & R_{s^+, \varepsilon}^\theta(t; x, y) \\ & = \delta \left(\Psi_\varepsilon^\theta(t) - S_\varepsilon(t-s) \Psi_\varepsilon^\theta(s) \right) (x, y) \\ & \quad - \theta(t, x) \left(X_\varepsilon(t) - S_\varepsilon(t-s) X_\varepsilon(s) \right) (x, y) \\ & = \int_s^t \int_{-\pi}^{\pi} \left(p_{t-r}^\varepsilon(y-z) - p_{t-r}^\varepsilon(x-z) \right) \left(\theta(r, z) - \theta(t, x) \right) H_\varepsilon dW(r, z), \end{aligned} \tag{3.47}$$

and

$$R_{s^-, \varepsilon}^\theta(t) = \delta \left(S_\varepsilon(t-s) \Psi_\varepsilon^\theta(s) \right) (x, y) - \theta(t, x) \left(S_\varepsilon(t-s) X_\varepsilon(s) \right) (x, y). \tag{3.48}$$

We assume that we control the space-time regularity of θ for $t > s$. Then we can obtain a regularity result for $R_{s^+, \varepsilon}^\theta(t; x, y)$ using Lemma 3.5. To this end, recalling the definition of the parabolic Hölder norms $\|\cdot\|_{C_{[\varepsilon^2, t]}^{\alpha/2, \alpha}}$ and $\|\cdot\|_{C_t^{\alpha/2, \alpha}}$ in (1.14) and (1.15), for $K > 0$ we introduce the stopping time

$$\varrho_{\varepsilon, K}^X = \inf \left\{ t \geq 0: \|X\|_{C_t^{\alpha/2, \alpha}} \geq K \text{ or } \|X_\varepsilon - X_0\|_{C_t^{\alpha/2, \alpha}} \geq 1 \right\}. \tag{3.49}$$

Observe (recalling the definitions (2.20a) and (2.22a)) that $\varrho_{\varepsilon, K}^X \geq \sigma_K^X \wedge \varrho_\varepsilon^X$ almost surely.

Lemma 3.6. *Suppose $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Then let $\alpha_1, \alpha_2 > 0$ and $p \geq 1$ satisfy*

$$\alpha_1 < \frac{\alpha \lambda_1}{2} - \frac{1}{p}, \quad \alpha_2 < \frac{(1+2\alpha)\lambda_2}{2} - \frac{1}{p}$$

for some $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$. Then for any $s \in [0, T]$ and any stopping time τ that almost surely satisfies

$$s \leq \tau \leq \varrho_{\varepsilon, K}^X \wedge T \tag{3.50}$$

we get the following bounds:

$$\mathbb{E} \|R_{s^+, \varepsilon}^\theta\|_{C^{\alpha_1}([s, \tau], \mathcal{B}^{\alpha_2})}^p \lesssim (1 + K^p) \mathbb{E} \|\theta\|_{C_{[s, \tau]}^{\alpha/2, \alpha}}^p,$$

$$\mathbb{E} \|R_{s^+, \varepsilon}^\theta - R_{s^+}^\theta\|_{\mathcal{C}^{\alpha_1}([s, \tau], \mathcal{B}^{\alpha_2})}^p \lesssim \varepsilon^{(1-\lambda_1-\lambda_2)\alpha p} (1 + K^p) \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p.$$

As before, we give the version of Lemma 3.6 that we will actually use in a Corollary. Recall the definition of the norm $\|\cdot\|_{\mathcal{B}_{[s, t]}^{2\tilde{\alpha}}}$ given in (1.18).

Corollary 3.7. *Suppose that $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $\tilde{\alpha} \in (\frac{1}{3}, \frac{1+2\alpha}{4})$. Furthermore, assume that*

$$p > \frac{2 + 6\alpha}{\alpha(1 + 2\alpha - 4\tilde{\alpha})}.$$

Then for

$$\lambda < 1 - \frac{4\tilde{\alpha}}{1 + 2\alpha} - \frac{1}{p} \frac{2 + 6\alpha}{\alpha(1 + 2\alpha)},$$

for $s \in [0, T]$, and any stopping time τ that satisfies (3.50) we get

$$\mathbb{E} \|R_{s^+, \varepsilon}^\theta\|_{\mathcal{B}_{[s, \tau]}^{2\tilde{\alpha}}}^p \lesssim T^{\frac{\lambda\alpha}{2}} \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p, \quad (3.51)$$

and

$$\mathbb{E} \|R_{s^+, \varepsilon}^\theta - R_{s^+}^\theta\|_{\mathcal{B}_{[s, \tau]}^{2\tilde{\alpha}}}^p \lesssim \varepsilon^{\lambda\alpha} \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p. \quad (3.52)$$

Proof of Lemma 3.6. To shorten the notation we always write $t^\tau = t \wedge \tau$. The statement follows from Lemma B.3 as soon as we have established that for all $x_1, x_2 \in [-\pi, \pi]$, all $s < t$ and for $s < t_1 < t_2 \leq T$ we have the following bounds.

$$\mathbb{E} |R_{s^+, \varepsilon}^\theta(t^\tau; x_1, x_2)|^p \lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p |x_1 - x_2|^{\frac{(1+2\alpha)p}{2}}, \quad (3.53a)$$

$$\begin{aligned} \mathbb{E} |R_{s^+, \varepsilon}^\theta(t_2^\tau; x_1, x_2) - R_{s^+, \varepsilon}^\theta(t_1^\tau; x_1, x_2)|^p \\ \lesssim (1 + K^p) \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p (t_2 - t_1)^{\frac{\alpha p}{2}}, \end{aligned} \quad (3.53b)$$

$$\mathbb{E} |\delta R_{s^+, \varepsilon}^\theta(t^\tau)|_{[x_1, x_2]}^p \lesssim K^p \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p |x_1 - x_2|^{(\alpha + \frac{1}{2})p}, \quad (3.53c)$$

$$\mathbb{E} |R_{s^+, \varepsilon}^\theta(t^\tau; x_1, x_2) - R_{s^+}^\theta(t^\tau; x_1, x_2)|^p \lesssim \varepsilon^{\alpha p} (1 + K^p) \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha}}^p. \quad (3.53d)$$

We start by deriving (3.53a). Using BDG-equality we see that

$$\begin{aligned} \mathbb{E} |R_{s^+, \varepsilon}^\theta(t^\tau; x_1, x_2)|^p &\lesssim \mathbb{E} \left(\int_s^{t^\tau} \int_{-\pi}^{\pi} (p_{t^\tau-s}^\varepsilon(x_1 - y) - p_{t^\tau-s}^\varepsilon(x_2 - y))^2 |\theta(s, y) - \theta(t^\tau, x_1)|^2 dy ds \right)^{\frac{p}{2}} \\ &\lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p \left(\int_s^{t^\tau} \int_{-\pi}^{\pi} (p_{t-r}^\varepsilon(x_1 - y) - p_{t-r}^\varepsilon(x_2 - y))^2 \right. \\ &\quad \left. \times (|t - r|^\alpha + |y - x_1|^{2\alpha}) dy dr \right)^{\frac{p}{2}} \\ &\lesssim \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^{\alpha/2, \alpha}}^p |x_1 - x_2|^{1+2\alpha}. \end{aligned}$$

Here in the last line we have used Lemma 3.5.

In the derivation of the remaining bounds we use the notation

$$\hat{X}_\varepsilon(t) = X_\varepsilon(t) - S_\varepsilon(t - s)X_\varepsilon(s) \quad \text{and} \quad \hat{\Psi}_\varepsilon^\theta(t) = (\Psi_\varepsilon^\theta(t) - S_\varepsilon(t - s)\Psi_\varepsilon^\theta(s)).$$

To see (3.53b) write using the definition (3.47) of $R_{s^+, \varepsilon}^\theta$

$$\begin{aligned}
& \mathbb{E} |R_{s^+, \varepsilon}^\theta(t_1^\tau; x, y) - R_{s^+, \varepsilon}^\theta(t_2^\tau; x, y)|^p \\
& \lesssim \sup_{x \in [-\pi, \pi]} \mathbb{E} |\widehat{\Psi}_\varepsilon^\theta(t_1^\tau, x) - \widehat{\Psi}_\varepsilon^\theta(t_2^\tau, x)|^p \\
& \quad + \sup_{x, y \in [-\pi, \pi]} \mathbb{E} \left[|\theta(t_1^\tau, x) - \theta(t_2^\tau, x)|^p |\widehat{X}_\varepsilon(t_1^\tau, y)|^p \right] \\
& \quad + \sup_{x, y \in [-\pi, \pi]} \mathbb{E} \left[|\theta(t_2^\tau, x)|^p |\widehat{X}_\varepsilon(t_1^\tau, y) - \widehat{X}_\varepsilon(t_2^\tau, y)|^p \right] \\
& \lesssim (t_2 - t_1)^{\frac{\alpha p}{2}} \left(\mathbb{E} \|\theta\|_{\mathcal{C}_{[s, t^\tau]}^p}^p + K^p \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, t]}^{\alpha/2, \alpha}}^p + K^p \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, t^\tau]}^p}^p \right).
\end{aligned}$$

Here we have strongly used the a priori bound on the regularity of X_ε provided by the the stopping time $\varrho_{\varepsilon, K}^X$ as well as the fact that according to Lemma 6.4 the approximated heat semigroup is bounded from $C^\alpha \rightarrow C^0$. The bound (3.53c) can be derived in a similar way. Actually, using the definition of $R_{s^+, \varepsilon}^\theta$ (3.47) we get that

$$\delta R_{s^+, \varepsilon}^\theta(t; z_1, z_2, z_3) = -(\theta(t, z_1) - \theta(t, z_2)) (\widehat{X}_\varepsilon(t, z_2) - \widehat{X}_\varepsilon(t, z_3)).$$

Thus we get that

$$\mathbb{E} |\delta R_{s^+, \varepsilon}^\theta(t^\tau)|_{[x, y]}^p \lesssim |x - y|^{(\alpha + \frac{1}{2})p} K^p \mathbb{E} \|\theta\|_{\mathcal{C}_{[s, \tau]}^\alpha}^p.$$

To obtain (3.53d) we write

$$\begin{aligned}
\mathbb{E} |R_{s^+, \varepsilon}^\theta(t^\tau, x, y) - R_{s^+, \varepsilon}^\theta(t^\tau, x, y)|^p & \lesssim \sup_{x \in [-\pi, \pi]} \mathbb{E} |\widehat{\Psi}_\varepsilon^\theta(t^\tau, x) - \widehat{\Psi}_\varepsilon^\theta(t^\tau, x)|^p \\
& \quad + \sup_{x, y \in [-\pi, \pi]} \mathbb{E} \left[|\theta(t^\tau, x)|^p |\widehat{X}_\varepsilon(t^\tau, y) - \widehat{X}_\varepsilon(t^\tau, y)|^p \right].
\end{aligned}$$

So that (3.53d) follows from (3.16d) and the definition of the stopping time (3.49). This finishes the proof of (3.53). \square

If we do not control the temporal regularity of the θ we need to use the regularising property of the heat semigroup to get the desired 2α regularity. The price we have to pay is a blowup for at $t \downarrow s$.

Lemma 3.8. *Suppose that $\tilde{\alpha} \leq \alpha \in (\frac{1}{3}, \frac{1}{2})$ and $\tilde{\beta} = \tilde{\alpha} + \kappa$ for a small constant $\kappa > 0$. Furthermore, fix an $s \in [0, T]$. Then we get for any stopping time that satisfies (3.50) and any $p > \frac{6}{1-2\alpha}$*

$$\mathbb{E} \|R_{s^-, \varepsilon}^\theta\|_{\mathcal{B}_{[s, \tau], \tilde{\beta}}^{2\tilde{\alpha}}}^p \lesssim T^{\alpha - \tilde{\alpha} + \kappa/2} \mathbb{E} \|\theta\|_{\mathcal{C}_\tau}^p.$$

Proof. According to (3.48) we have

$$R_{s^-, \varepsilon}^\theta(t) = \delta \left(S_\varepsilon(t-s) \Psi_\varepsilon^\theta(s) \right) - \theta(t, \cdot) (S_\varepsilon(t-s) X_\varepsilon(s))$$

the result follows immediately from the regularising property of the approximated heat semigroup (Lemma 6.4) as well as the regularity properties of $\Psi_\varepsilon^\theta, X_\varepsilon$ from Corollaries 3.2 and 3.4. \square

4. THE REACTION TERM

In this section we derive bounds for the convergence of the reaction term.

4.1. The Gaussian process and stochastic fluctuations. As before, we consider the solution to the approximate stochastic heat equation

$$X^\varepsilon(t, x) = \sum_{k \in \mathbb{Z}} q_\varepsilon^k \xi_\varepsilon^k(t) e^{ikx}, \quad (4.1)$$

together with its area process $\mathbf{X}^\varepsilon(t; x, y)$, where we use the notation from Section 3. We recall in particular that

$$\mathbb{E}[\xi_\varepsilon^k(s) \otimes \xi_\varepsilon^l(t)] = \delta_{k, -l} \mathcal{K}_k^{s, t} \text{Id},$$

where

$$\mathcal{K}_k^{s, t} = \begin{cases} e^{-f(\varepsilon k)k^2|t-s|} - e^{-f(\varepsilon k)k^2(t+s)}, & k \neq 0, \\ s \wedge t, & k = 0. \end{cases}$$

Recall also that

$$\Lambda = \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{(1 - \cos(yt))h^2(t)}{t^2 f(t)} \mu(dy) dt.$$

The goal of this section is to prove the following result, which yields a sharp bound, uniform in time, on the difference

$$D_\varepsilon \mathbf{X}_\varepsilon(t, \cdot) - \Lambda \text{Id}$$

measured in the spatial negative Sobolev norm $|u|_{H^{-\alpha}}$ for $\alpha < \frac{1}{2}$. For this purpose we introduce for fixed $z \in \mathbb{R}$ the quantity

$$\Lambda_{z, \varepsilon}(t) = \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} (q_\varepsilon^k)^2 \mathcal{K}_k^{t, t} \sin^2\left(\frac{k\varepsilon z}{2}\right) \quad (4.2)$$

as well as the integrated version

$$\Lambda_\varepsilon(t) = \int_{\mathbb{R}} \Lambda_{z, \varepsilon}(t) \mu(dz).$$

Proposition 4.1. *Let $1 \leq p < \infty$ and $0 < \alpha < \frac{1}{2}$. For all $\kappa > 0$ sufficiently small, we have*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| D_\varepsilon \mathbf{X}_\varepsilon(t, \cdot) - \Lambda_\varepsilon(t) \text{Id} \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} \lesssim \varepsilon^{\alpha - \kappa}, \quad (4.3)$$

$$\sup_{t \in [0, T]} t^{\frac{\alpha}{2}} |\Lambda_\varepsilon(t) - \Lambda| \lesssim \varepsilon^{\alpha - \kappa}. \quad (4.4)$$

Remark 4.2. The term $\Lambda_\varepsilon(t) - \Lambda$ appears due to the fact that the process X^ε starts at 0. If we had considered the stationary version of the process X^ε as in [HM10], then it would not have been necessary to estimate this term.

Proof of Proposition 4.1. It follows from Lemma 4.3 and Lemma 4.4 below that

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| D_\varepsilon \mathbf{X}_\varepsilon(t, \cdot) - \Lambda_\varepsilon(t) \text{Id} \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} \\ & \lesssim \int_{\mathbb{R}} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \frac{1}{\varepsilon} \mathbf{X}_\varepsilon(t; \cdot, \cdot + \varepsilon z) - \Lambda_{z, \varepsilon}(t) \text{Id} \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} |\mu|(dz) \end{aligned}$$

$$\lesssim \varepsilon^{\alpha-\kappa} \int_{\mathbb{R}} |z|^{1+\alpha-\kappa} |\mu|(dz) \lesssim \varepsilon^{\alpha-\kappa},$$

which proves (4.3).

In order to prove (4.4) we claim that for any $t \in [0, T]$ and $\kappa > 0$,

$$|\Lambda_{z,\varepsilon}(t) - \Lambda_z| \lesssim |z|^2 (\varepsilon^{\alpha-\kappa} t^{-\frac{\alpha}{2}} + \varepsilon). \quad (4.5)$$

Integrating this inequality with respect to z and using the fact that the second moment of $|\mu|$ is finite, we obtain the desired estimate.

It remains to prove the claim (4.5). For this purpose we consider the quantity

$$\tilde{\Lambda}_{z,\varepsilon} = \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} (q_\varepsilon^k)^2 \sin^2 \left(\frac{k\varepsilon z}{2} \right)$$

and estimate for $\lambda \in [0, 1]$ and $p = \lambda - \frac{1}{2} - \kappa$,

$$\begin{aligned} |\Lambda_{z,\varepsilon}(t) - \tilde{\Lambda}_{z,\varepsilon}| &\leq \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} (q_\varepsilon^k)^2 |\mathcal{K}_k^{t,t} - 1| \sin^2 \left(\frac{k\varepsilon z}{2} \right) \\ &\lesssim \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} \frac{1}{k^2 f(\varepsilon k)} e^{-2k^2 f(\varepsilon k)t} \left(1 \wedge (k\varepsilon z)^2 \right) \\ &\lesssim \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} \frac{1}{k^2} \frac{1}{k^{2p} t^p} (k\varepsilon z)^{2\lambda} \lesssim \varepsilon^{2\lambda-1} |z|^{2\lambda} t^{-(\lambda-\frac{1}{2}-\kappa)}. \end{aligned} \quad (4.6)$$

Furthermore, as in [HM10, Proposition 4.6] we use the fact that for any function $g : \mathbb{R} \rightarrow \mathbb{R}$ of bounded variation,

$$\left| \sum_{k \in \mathbb{R}} \varepsilon g(\varepsilon k) - \int_{\mathbb{R}} g(t) dt \right| \leq \varepsilon |g|_{\text{BV}},$$

Using this fact together with the assumptions on h and f , we obtain

$$\begin{aligned} |\Lambda_z - \tilde{\Lambda}_{z,\varepsilon}| &\lesssim \varepsilon \left| s \mapsto \frac{h^2(s)}{f(s)} \frac{\sin^2(\frac{sz}{2})}{s^2} \right|_{\text{BV}} \\ &\lesssim \varepsilon \left| \frac{h^2}{f} \right|_{L^\infty} \left| s \mapsto \frac{\sin^2(\frac{sz}{2})}{s^2} \right|_{\text{BV}} + \left| \frac{h^2}{f} \right|_{\text{BV}} \left| s \mapsto \frac{\sin^2(\frac{sz}{2})}{s^2} \right|_{L^\infty} \\ &\lesssim \varepsilon |z|^2. \end{aligned} \quad (4.7)$$

Combining the estimates (4.6) (with $\lambda = \frac{1}{2}(1+\alpha-\kappa)$) and (4.7), the claim follows and the proof is complete. \square

For a matrix A , it will be convenient to work with the decomposition $A = A^+ + A^-$, where

$$[A^\pm]_{ij} := \frac{1}{2} (A_{ij} \pm A_{ji}).$$

The following two lemmas are the main ingredients in the proof of (4.3).

Lemma 4.3. *Let $1 \leq p < \infty$ and $0 < \alpha < \frac{1}{2}$. For all $\kappa > 0$ and $z \in \mathbb{R}$ we have*

$$\mathbb{E} \left[\sup_{t \geq 0} \left| \frac{1}{\varepsilon} \mathbf{X}_\varepsilon^-(t; \cdot, \cdot + \varepsilon z) \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} \lesssim \varepsilon^{\alpha-\kappa} |z|^{1+\alpha-\kappa}. \quad (4.8)$$

The proof of this result is a rather delicate computation, in which logarithmic divergences appear if one is not careful. In particular, in the calculation below it is important to restrict the sum to non-diagonal terms.

Proof. Note that

$$\frac{1}{\varepsilon} \mathbf{X}_\varepsilon(t; x, x + \varepsilon z) = i \sum_{k, l \in \mathbb{Z}} l q_\varepsilon^k q_\varepsilon^l I_{z, kl} \left(\xi_k(t) \otimes \xi_l(t) \right) e^{i(k+l)x},$$

where

$$I_{z, kl} := \frac{1}{\varepsilon} \int_x^{x+\varepsilon z} (e^{ikw} - e^{ikx}) e^{ilw} dw = \frac{1}{i\varepsilon} \left(\frac{1 - e^{il\varepsilon z}}{l} - \frac{1 - e^{i(k+l)\varepsilon z}}{k+l} \right),$$

the latter expression being valid whenever $l, k+l \neq 0$. As a consequence, we have the identity

$$\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^-(t; x, x + \varepsilon z) = \sum_{k, l \in \mathbb{Z}} q_\varepsilon^k q_\varepsilon^l J_{z, kl}^- \xi_k(t) \otimes \xi_l(t) e^{i(k+l)x},$$

where

$$J_{z, kl}^- := \frac{i}{2} (l I_{z, kl} - k I_{z, lk}) = \frac{k-l}{2\varepsilon} \left(\frac{1 - e^{i(k+l)\varepsilon z}}{k+l} + \frac{e^{ik\varepsilon z} - e^{il\varepsilon z}}{k-l} \right),$$

and the latter formula holds if $|k| \neq |l|$. Writing $\zeta_{k,l}^{i,j}(t) := \xi_k^i(t) \xi_l^j(t)$ for brevity we obtain

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{\varepsilon} \left(\mathbf{X}_\varepsilon^-(t; \cdot, \cdot + \varepsilon z) - \mathbf{X}_\varepsilon^-(s; \cdot, \cdot + \varepsilon z) \right) \right|_{H^{-\alpha}}^2 \\ &= \sum_{k, l, m \in \mathbb{Z}} (1 + m^2)^{-\alpha} q_\varepsilon^k q_\varepsilon^{m-k} q_\varepsilon^l q_\varepsilon^{m-l} J_{z, k, m-k}^- \overline{J_{z, l, m-l}^-} \\ & \quad \times \mathbb{E} \left[\sum_{i \neq j} \left(\zeta_{k, m-k}^{i,j}(t) - \zeta_{k, m-k}^{i,j}(s) \right) \left(\zeta_{-l, -(m-l)}^{i,j}(t) - \zeta_{-l, -(m-l)}^{i,j}(s) \right) \right]. \end{aligned}$$

Note that we sum over $i \neq j$, since the diagonal terms are 0. For $i \neq j$, we use the fact that

$$\begin{aligned} \mathbb{E} \left[\zeta_{k, m-k}^{i,j}(s) \zeta_{-l, -(m-l)}^{i,j}(t) \right] &= \mathbb{E} \left[\xi_k^i(s) \xi_{-l}^i(t) \right] \mathbb{E} \left[\xi_{m-k}^j(s) \xi_{-(m-l)}^j(t) \right] \\ &= \delta_{k,l} \mathcal{K}_k^{s,t} \mathcal{K}_{m-k}^{s,t}, \end{aligned}$$

to obtain, for $\kappa \in [0, \frac{1}{2}]$,

$$\begin{aligned} & \mathbb{E} \left[\zeta_{k, m-k}^{i,j}(t) - \zeta_{k, m-k}^{i,j}(s) \right] \left[\zeta_{-l, -(m-l)}^{i,j}(t) - \zeta_{-l, -(m-l)}^{i,j}(s) \right] \\ &= \delta_{k,l} \left(\mathcal{K}_k^{t,t} \mathcal{K}_{m-k}^{t,t} - 2 \mathcal{K}_k^{s,t} \mathcal{K}_{m-k}^{s,t} + \mathcal{K}_k^{s,s} \mathcal{K}_{m-k}^{s,s} \right) \\ &\lesssim \delta_{k,l} \left(|1 - e^{-f(\varepsilon k)k^2|t-s|}| + |1 - e^{-f(\varepsilon(m-k))(m-k)^2|t-s|}| \right) \\ &\lesssim \delta_{k,l} \left(f(\varepsilon k)^\kappa |k|^{2\kappa} + f(\varepsilon(m-k))^\kappa |m-k|^{2\kappa} \right) |t-s|^\kappa. \end{aligned}$$

Furthermore, we observe that

$$|J_{z, kl}^-|^2 = \left((k-l)z \left(S((k+l)\varepsilon z) - S((k-l)\varepsilon z) \right) \right)^2,$$

where $S(x) = \sin(x/2)/x$. Using the assumption that h is bounded, we obtain for $k, l \neq 0$,

$$q_\varepsilon^k q_\varepsilon^l \lesssim \frac{1}{\sqrt{f(\varepsilon k) f(\varepsilon l)} |kl|} \lesssim \frac{1}{\sqrt{f(\varepsilon k) f(\varepsilon l)} |(k+l)^2 - (k-l)^2|}.$$

Since $J_{k,l} = 0$ if $k = 0$ or $l = 0$, and moreover $f \geq c_f$, it follows that

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{\varepsilon} \left(\mathbf{X}_\varepsilon^-(t; \cdot, \cdot + \varepsilon z) - \mathbf{X}_\varepsilon^-(s; \cdot, \cdot + \varepsilon z) \right) \right|_{H^{-\alpha}}^2 \\ & \lesssim \sum_{k, m \in \mathbb{Z}} (1 + m^2)^{-\alpha} (q_\varepsilon^k q_\varepsilon^{m-k})^2 |J_{z, k, m-k}^-|^2 \\ & \quad \times \left(f(\varepsilon k)^\kappa |k|^{2\kappa} + f(\varepsilon(m-k))^\kappa |m-k|^{2\kappa} \right) |t-s|^\kappa \\ & \lesssim \sum_{k, l \in \mathbb{Z}} (1 + (k+l)^2)^{-\alpha} (q_\varepsilon^k q_\varepsilon^l)^2 |J_{z, k, l}^-|^2 \left(f(\varepsilon k)^\kappa |k|^{2\kappa} + f(\varepsilon l)^\kappa |l|^{2\kappa} \right) |t-s|^\kappa \\ & \lesssim |t-s|^\kappa |z|^2 \sum_{k, l \neq 0} (1 + (k+l)^2)^{-\alpha} |k-l|^2 \left| \frac{S((k+l)\varepsilon z) - S((k-l)\varepsilon z)}{(k+l)^2 - (k-l)^2} \right|^2 \\ & \quad \times \left(|k+l|^{2\kappa} + |k-l|^{2\kappa} \right) \\ & \lesssim |t-s|^\kappa |z|^2 \sum_{|k| \neq |l|} (1 + k^2)^{-\alpha} l^2 \left| \frac{S(k\varepsilon z) - S(l\varepsilon z)}{k^2 - l^2} \right|^2 \left(|k|^{2\kappa} + |l|^{2\kappa} \right). \end{aligned}$$

where we used the change of variables $(k+l, k-l) \rightsquigarrow (k, l)$. We infer that

$$\begin{aligned} & \mathbb{E} \left| \frac{1}{\varepsilon} \left(\mathbf{X}_\varepsilon^-(t; \cdot, \cdot + \varepsilon z) - \mathbf{X}_\varepsilon^-(s; \cdot, \cdot + \varepsilon z) \right) \right|_{H^{-\alpha}}^2 \\ & \lesssim \varepsilon^{2\alpha-2\kappa} z^2 \int_{\mathbb{R}^2} \left(\frac{x}{y^\alpha} \frac{S(xz) - S(yz)}{x^2 - y^2} \right)^2 (x^2 + y^2)^\kappa dx dy \\ & \asymp \varepsilon^{2\alpha-2\kappa} |z|^{2+2\alpha-2\kappa} \int_{\mathbb{R}^2} \left(\frac{x}{y^\alpha} \frac{S(x) - S(y)}{x^2 - y^2} \right)^2 (x^2 + y^2)^\kappa dx dy. \end{aligned}$$

Taking into account that $\alpha < \frac{1}{2}$ and using the estimate

$$\frac{|S(x) - S(y)|}{|x^2 - y^2|} \lesssim 1 \wedge \frac{1}{x^2 + y^2},$$

we infer that

$$\mathbb{E} \left| \frac{1}{\varepsilon} \left(\mathbf{X}_\varepsilon^-(t; \cdot, \cdot + \varepsilon z) - \mathbf{X}_\varepsilon^-(s; \cdot, \cdot + \varepsilon z) \right) \right|_{H^{-\alpha}}^2 \lesssim \varepsilon^{2\alpha-2\kappa} |z|^{2+2\alpha-2\kappa}. \quad (4.9)$$

Since $\mathbf{X}_\varepsilon^-(0) = 0$, the result follows from Lemma B.4. Note that (B.20) is satisfied since $\mathbf{X}_\varepsilon^-(t, \cdot)$ belongs to the $H^{-\alpha}$ -valued second order Wiener chaos, see e.g., [Nua06]. \square

The next result provides the corresponding estimate for the symmetric part of \mathbf{X}_ε .

Lemma 4.4. *Let $1 \leq p < \infty$ and $0 < \alpha < \frac{1}{2}$. For any $z \in \mathbb{R}$ and $\kappa > 0$ we have*

$$\mathbb{E} \left[\sup_{t \geq 0} \left| \frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(t; \cdot, \cdot + \varepsilon z) - \Lambda_{z, \varepsilon}(t) \text{Id} \right|_{H^{-\alpha}}^p \right]^{\frac{1}{p}} \lesssim \varepsilon^{\alpha-\kappa} |z|^{1+\alpha-\kappa}. \quad (4.10)$$

Proof. As in the proof of Lemma 4.3, we write

$$\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(t; x, x + \varepsilon z) = \sum_{k, l \in \mathbb{Z}} q_\varepsilon^k q_\varepsilon^l J_{z, kl}^+ \xi_k(t) \otimes \xi_l(t) e^{i(k+l)x},$$

where

$$J_{z, kl}^+ := \frac{i}{2} (lI_{z, kl} + kI_{z, lk}) = \frac{1}{2\varepsilon} (1 - e^{ik\varepsilon z}) (1 - e^{il\varepsilon z}),$$

and $I_{z, kl}$ is as in the proof of Lemma 4.3. Note that

$$\Lambda_{z, \varepsilon}(t) = \mathbb{E} \left[\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(t; x, x + \varepsilon z) \right] = \sum_k |q_\varepsilon^k|^2 J_{z, k, -k}^+ \mathcal{K}_k^{t, t}.$$

Writing

$$\tilde{\zeta}_{k, l}(t) = \xi_k(t) \otimes \xi_l(t) - \mathbb{E}[\xi_k(t) \otimes \xi_l(t)]$$

for brevity, we estimate

$$\begin{aligned} & \mathbb{E} \left| \left(\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(t; x, x + \varepsilon z) - \Lambda_{z, \varepsilon}(t) I \right) - \left(\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(s; x, x + \varepsilon z) - \Lambda_{z, \varepsilon}(s) I \right) \right|_{H^{-\alpha}}^2 \\ &= \sum_{k, l, m \in \mathbb{Z}} (1 + m^2)^{-\alpha} q_\varepsilon^k q_\varepsilon^{m-k} q_\varepsilon^l q_\varepsilon^{m-l} \overline{J_{z, k, m-k}^+ J_{z, l, m-l}^+} \\ & \quad \times \mathbb{E} \left[\text{tr} \left([\tilde{\zeta}_{k, m-k}(t) - \tilde{\zeta}_{k, m-k}(s)] [\tilde{\zeta}_{l, m-l}(t) - \tilde{\zeta}_{l, m-l}(s)]^* \right) \right]. \end{aligned}$$

A case distinction argument yields

$$\begin{aligned} & \mathbb{E} \left[\text{tr} \left([\tilde{\zeta}_{k, m-k}(t) - \tilde{\zeta}_{k, m-k}(s)] [\tilde{\zeta}_{l, m-l}(t) - \tilde{\zeta}_{l, m-l}(s)]^* \right) \right] \\ &= \left(n^2 \delta_{k+l, m} + n \delta_{k, l} \right) \left(\mathcal{K}_k^{t, t} \mathcal{K}_{m-k}^{t, t} - 2 \mathcal{K}_k^{s, t} \mathcal{K}_{m-k}^{s, t} + \mathcal{K}_k^{s, s} \mathcal{K}_{m-k}^{s, s} \right), \end{aligned}$$

and using the definition of \mathcal{K} we infer that

$$\begin{aligned} & \left| \mathcal{K}_k^{t, t} \mathcal{K}_{m-k}^{t, t} - 2 \mathcal{K}_k^{s, t} \mathcal{K}_{m-k}^{s, t} + \mathcal{K}_k^{s, s} \mathcal{K}_{m-k}^{s, s} \right| \\ & \lesssim \left| 1 - e^{-f(\varepsilon k)k^2|t-s|} \right| + \left| 1 - e^{-f(\varepsilon(m-k))(m-k)^2|t-s|} \right| \\ & \lesssim \left(f(\varepsilon k)|k|^{2\kappa} + f(\varepsilon(m-k))|m-k|^{2\kappa} \right) |t-s|^\kappa \end{aligned}$$

for $\kappa \in [0, \frac{1}{2})$. Using the estimate $|q_k| \lesssim \frac{1}{\sqrt{f(\varepsilon k)|k|}}$ for $k \neq 0$, together with the bound

$$|J_{z, kl}^+|^2 \lesssim \frac{1 - \cos(k\varepsilon z)}{\varepsilon} \frac{1 - \cos(l\varepsilon z)}{\varepsilon},$$

we obtain

$$\begin{aligned} & \mathbb{E} \left| \left(\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(t; x, x + \varepsilon z) - \Lambda_{z, \varepsilon}(t) I \right) - \left(\frac{1}{\varepsilon} \mathbf{X}_\varepsilon^+(s; x, x + \varepsilon z) - \Lambda_{z, \varepsilon}(s) I \right) \right|_{H^{-\alpha}}^2 \\ & \lesssim \sum_{k, (m-k) \neq 0} (1 + m^2)^{-\alpha} (q_\varepsilon^k q_\varepsilon^{m-k})^2 |J_{k, m-k}^+|^2 \\ & \quad \times \left(\mathcal{K}_k^{t, t} \mathcal{K}_{m-k}^{t, t} - 2 \mathcal{K}_k^{s, t} \mathcal{K}_{m-k}^{s, t} + \mathcal{K}_k^{s, s} \mathcal{K}_{m-k}^{s, s} \right) \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k,l \neq 0} (1 + (k+l)^2)^{-\alpha} \frac{1 - \cos(k\varepsilon z)}{f(\varepsilon k)k^2\varepsilon} \frac{1 - \cos(l\varepsilon z)}{f(\varepsilon l)l^2\varepsilon} \\
&\quad \times \left(f(\varepsilon k)^\kappa |k|^{2\kappa} + f(\varepsilon l)^\kappa |l|^{2\kappa} \right) |t-s|^\kappa \\
&\lesssim \sum_{k,l \neq 0} \frac{1}{|k|^\alpha |l|^\alpha} \frac{1 - \cos(k\varepsilon z)}{k^2\varepsilon} \frac{1 - \cos(l\varepsilon z)}{l^2\varepsilon} \left(|k|^{2\kappa} + |l|^{2\kappa} \right) |t-s|^\kappa \\
&\lesssim |t-s|^\kappa \varepsilon^{2\alpha-2\kappa} |z|^{2+2\alpha-2\kappa} .
\end{aligned}$$

As in Lemma 4.3, the result now follows from Lemma B.4. \square

4.2. Bounds for the reaction term. With Proposition 4.1 at hand we will now derive the estimates for the reaction terms defined in Section 2.

Let $\varepsilon \in (0, 1)$. We fix a Hölder exponent $0 \leq \tilde{\alpha} < \alpha < \frac{1}{2}$ and fix functions $\varphi \in \mathcal{C}([0, T]; \mathcal{C}^\alpha)$ and $\varphi_\varepsilon = \varphi + \mathcal{D}_\varepsilon^{(1)} + \mathcal{D}_\varepsilon^{(2)}$ with $\mathcal{D}_\varepsilon^{(1)} \in \mathcal{C}([0, T]; \mathcal{C}^\alpha)$ and $\mathcal{D}_\varepsilon^{(2)} \in \mathcal{C}([0, T]; H^{-\alpha})$. We shall use the notation $\|u\|_{H_t^{-\alpha}}$ to denote the temporal supremum of the negative spatial Sobolev norm, i.e.

$$\|u\|_{H_t^{-\alpha}} = \sup_{s \in [0, t]} |u(s)|_{H^{-\alpha}} .$$

Furthermore, we consider functions $u, v, u_\varepsilon, v_\varepsilon \in L^\infty([0, T]; \mathcal{C}^\alpha)$. The reaction term naturally splits into two parts:

$$\Phi(t) = \int_0^t S(t-s)F(u(s)) ds , \quad \Upsilon(t) = \int_0^t S(t-s)\mathcal{F}(u(s), v(s)) ds ,$$

where

$$\mathcal{F}^i(u, v) = \partial_k G_j^i(u) v_l^k \varphi^{l,m} v_m^j .$$

Similarly, for $\varepsilon > 0$ we define

$$\Phi_\varepsilon(t) = \int_0^t S_\varepsilon(t-s)F(u_\varepsilon(s)) ds , \quad \Upsilon_\varepsilon(t) = \int_0^t S_\varepsilon(t-s)\mathcal{F}_\varepsilon(u_\varepsilon(s), v_\varepsilon(s)) ds ,$$

where

$$\mathcal{F}_\varepsilon^i(u, v) = \partial_k G_j^i(u) v_l^k \varphi_\varepsilon^{l,m} v_m^j .$$

In order to deal with blowup at 0 we shall consider the weighted norms $\|u\|_{\mathcal{C}_{t,\beta}^\alpha}$ defined in (1.16). Throughout the remainder of this section we assume that the norms

$$\|\varphi\|_{\mathcal{C}_T^\alpha} , \quad \|u\|_{\mathcal{C}_T^\alpha} , \quad \|v\|_{\mathcal{C}_T^\alpha} , \quad \|u_\varepsilon\|_{\mathcal{C}_T^\alpha} , \quad \|u'_\varepsilon\|_{\mathcal{C}_T^\alpha} ,$$

are bounded by a constant $K > 0$, which does not depend on ε .

Remark 4.5. In our application, u and u_ε will be as in the previous sections, and

$$\begin{aligned}
v &= u' , \quad v_\varepsilon = u'_\varepsilon , \quad \varphi = \Lambda \text{Id} , \\
\mathcal{D}_\varepsilon^{(1)} &= (\Lambda_\varepsilon(t) - \Lambda) \text{Id} , \quad \mathcal{D}_\varepsilon^{(2)} = D_\varepsilon \mathbf{X}_\varepsilon(t, \cdot) - \Lambda_\varepsilon(t) \text{Id} .
\end{aligned}$$

We shall first prove a bound on the difference between Φ and Φ_ε .

Proposition 4.6. *Let $\tilde{\alpha} \leq \gamma < 1$. Then, for any $t \in [0, T]$ and $\kappa > 0$ we have*

$$\begin{aligned} |\Phi(t) - \Phi_\varepsilon(t)|_{C^\gamma} &\lesssim t^{1-\frac{1}{2}(\gamma-\tilde{\alpha})} \|u - u_\varepsilon\|_{C_t^{\tilde{\alpha}}} + \varepsilon^{1-\kappa}, \\ \|\Phi - \Phi_\varepsilon\|_{C^{\frac{\gamma}{2}}([0, T], \mathcal{C})} &\lesssim t^{1-\frac{\gamma}{2}-\kappa} \|u - u_\varepsilon\|_{C_t^{\tilde{\alpha}}} + \varepsilon^{1-\frac{\gamma}{2}-\kappa}, \end{aligned}$$

with implied constant depending on K .

Proof. Using Lemma 6.5 we obtain

$$\begin{aligned} &|\Phi(t) - \Phi_\varepsilon(t)|_{C^\gamma} \\ &\leq \left| \int_0^t S(t-s)(F(u) - F(u_\varepsilon))(s) ds \right|_{C^\gamma} + \left| \int_0^t (S - S_\varepsilon)(t-s)F(u_\varepsilon)(s) ds \right|_{C^\gamma} \\ &\lesssim t^{1-\frac{1}{2}(\gamma-\tilde{\alpha})} \|F(u) - F(u_\varepsilon)\|_{C_t^{\tilde{\alpha}}} + \varepsilon^{1-\kappa} t^{1-\frac{1}{2}(\gamma-\alpha+1)} \|F(u_\varepsilon)\|_{C_t^\alpha} \\ &\lesssim t^{1-\frac{1}{2}(\gamma-\tilde{\alpha})} \|u - u_\varepsilon\|_{C_t^{\tilde{\alpha}}} + \varepsilon^{1-\kappa} t^{1-\frac{1}{2}(\gamma-\alpha+1)}, \end{aligned}$$

which proves the first bound.

To prove the second inequality, we write

$$\begin{aligned} &(\Phi - \Phi_\varepsilon)(t) - (\Phi - \Phi_\varepsilon)(s) \\ &= \int_0^s \left((S - S_\varepsilon)(t-r) - (S - S_\varepsilon)(s-r) \right) F(u_\varepsilon(r)) dr \\ &\quad + \int_s^t (S - S_\varepsilon)(t-r) F(u_\varepsilon(r)) dr \\ &\quad + \int_0^s (S(t-r) - S(s-r)) (F(u(r)) - F(u_\varepsilon(r))) dr \\ &\quad + \int_s^t S(t-r) (F(u(r)) - F(u_\varepsilon(r))) dr \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We shall estimate the first term in two ways. First, using (6.18) and the fact that $|(S - S_\varepsilon)(t-s)|_{C^\alpha \rightarrow \mathcal{C}} \lesssim 1$,

$$\begin{aligned} |I_1|_{\mathcal{C}} &\lesssim \left| \int_0^s \left((S - S_\varepsilon)(t-r) - (S - S_\varepsilon)(s-r) \right) F(u_\varepsilon(r)) dr \right|_{\mathcal{C}} \\ &\lesssim \int_0^s |(S - S_\varepsilon)(s-r)|_{C^\alpha \rightarrow \mathcal{C}} |F(u_\varepsilon(r))|_{\mathcal{C}^\alpha} dr \\ &\lesssim \varepsilon^{1-\kappa} \int_0^s (s-r)^{-\frac{1}{2}(1-\alpha)} \|u_\varepsilon\|_{C_t^\alpha} dr \lesssim \varepsilon^{1-\kappa} \|u_\varepsilon\|_{C_t^\alpha}. \end{aligned} \tag{4.11}$$

On the other hand, we have the bound

$$\begin{aligned} |I_1|_{\mathcal{C}} &\lesssim \int_0^s \left(|S(t-s) - \text{Id}|_{C^{2-2\kappa} \rightarrow \mathcal{C}} |S(s-r)|_{C^\alpha \rightarrow C^{2-2\kappa}} \right. \\ &\quad \left. + |S_\varepsilon(t-s) - \text{Id}|_{C^{2-2\kappa} \rightarrow \mathcal{C}} |S_\varepsilon(s-r)|_{C^\alpha \rightarrow C^{2-2\kappa}} \right) |F(u_\varepsilon(r))|_{\mathcal{C}^\alpha} dr \\ &\lesssim \int_0^s (t-s)^{1-\kappa} (s-r)^{-\frac{1}{2}(2-\alpha-2\kappa)} |F(u_\varepsilon(r))|_{\mathcal{C}^\alpha} dr \lesssim (t-s)^{1-\kappa} \|u_\varepsilon\|_{C_t^\alpha}. \end{aligned} \tag{4.12}$$

Combining both estimates we infer that for $\gamma \in [0, 1]$,

$$\begin{aligned} |I_1|_{\mathcal{C}} &\lesssim \varepsilon^{1-\frac{\gamma}{2}-\kappa} (t-s)^{\frac{\gamma}{2}} \|u_\varepsilon\|_{\mathcal{C}_t^\alpha} \\ &\lesssim \varepsilon^{1-\frac{\gamma}{2}-\kappa} (t-s)^{\frac{\gamma}{2}} . \end{aligned} \quad (4.13)$$

Using (6.18) once more, the term I_2 is bounded by

$$\begin{aligned} |I_2|_{\mathcal{C}} &\lesssim \int_0^{t-s} |(S - S_\varepsilon)(r)|_{\mathcal{C}^\alpha \rightarrow \mathcal{C}} \|F(u_\varepsilon)\|_{\mathcal{C}_t^\alpha} dr \\ &\lesssim \varepsilon^{1-\kappa} \int_0^{t-s} r^{-\frac{1}{2}(1-\alpha)} \|u_\varepsilon\|_{\mathcal{C}_t^\alpha} dr \lesssim \varepsilon^{1-\kappa} (t-s)^{\frac{1}{2}(1+\alpha)} . \end{aligned} \quad (4.14)$$

To bound I_3 we proceed as in (4.11) to infer that

$$\begin{aligned} |I_3|_{\mathcal{C}} &\leq \int_0^s (S(t-r) - S(s-r)) (F(u(r)) - F(u_\varepsilon(r))) dr \\ &\lesssim \int_0^s (t-s)^{1-\kappa} (s-r)^{-\frac{1}{2}(2-\tilde{\alpha}-2\kappa)} \|F(u) - F(u_\varepsilon)\|_{\mathcal{C}_t^{\tilde{\alpha}}} dr \\ &\lesssim (t-s)^{1-\kappa} \|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \lesssim (t-s)^{\frac{\gamma}{2}} t^{1-\frac{\gamma}{2}} \|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} . \end{aligned}$$

The next term can be bounded brutally by

$$\begin{aligned} |I_4|_{\mathcal{C}} &\leq |t-s| \|F(u) - F(u_\varepsilon)\|_{\mathcal{C}_t^{\tilde{\alpha}}} \lesssim |t-s| \|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \\ &\lesssim (t-s)^{\frac{\gamma}{2}} t^{1-\frac{\gamma}{2}-\kappa} \|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} . \end{aligned}$$

Combining all of these estimates, we obtain the desired bound. \square

The following result gives a bound on the difference between Υ and Υ_ε .

Proposition 4.7. *Let $\tilde{\alpha} < \gamma < 1$. Then, for any $t \in [0, T]$ and $\kappa > 0$ we have*

$$\begin{aligned} |\Upsilon_\varepsilon(t) - \Upsilon(t)|_{\mathcal{C}^\gamma} &\lesssim \varepsilon^{1-\kappa} + t^{1-\frac{\gamma}{2}} \|\mathcal{D}_\varepsilon^{(1)}\|_{\mathcal{C}_{T, \frac{\gamma}{2}}^\alpha} + t^{\frac{1-\alpha-\gamma}{2}} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_T^{-\alpha}} \\ &\quad + t^{1-\frac{1}{2}(\gamma-\alpha+1)} (\|u - u_\varepsilon\|_{\mathcal{C}_T^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_T^{\tilde{\alpha}}}) , \\ \|\Upsilon_\varepsilon - \Upsilon\|_{\mathcal{C}^{\frac{\gamma}{2}}([0, T], \mathcal{C})} &\lesssim \varepsilon^{1-\frac{\gamma}{2}-\kappa} + T^{1-\frac{\gamma}{2}-\kappa} (\|u - u_\varepsilon\|_{\mathcal{C}_T^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_T^{\tilde{\alpha}}}) \\ &\quad + (T^{1-\frac{1}{2}(\alpha+\gamma)-\kappa} + \varepsilon^{1-\frac{\gamma}{2}-\kappa} T^\alpha) \|\mathcal{D}_\varepsilon^{(1)}\|_{H_T^{-\alpha}} \\ &\quad + T^{\frac{1}{2}(1-\alpha-\gamma)} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_t^{-\alpha}} . \end{aligned}$$

with implied constant depending on K .

Proof. We rewrite the difference $\Upsilon_\varepsilon - \Upsilon$ as

$$\begin{aligned} \Upsilon_\varepsilon(t) - \Upsilon(t) &= \int_0^t S_\varepsilon(t-s) (\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) - \mathcal{F}(u_\varepsilon, v_\varepsilon))(s) ds \\ &\quad + \int_0^t S_\varepsilon(t-s) (\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon))(s) ds \\ &\quad + \int_0^t (S - S_\varepsilon)(t-s) \mathcal{F}(u, v)(s) ds . \\ &=: I_1 + I_2 + I_3 . \end{aligned}$$

Using Lemma 4.8 we obtain

$$\begin{aligned}
|I_1|_{C^\gamma} &\leq \left| \int_0^t S_\varepsilon(t-s) \left(DG(u_\varepsilon) v_\varepsilon \mathcal{D}_\varepsilon^{(1)} v_\varepsilon \right)(s) ds \right|_{C^\gamma} \\
&\quad + \left| \int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) \left(DG(u_\varepsilon) v_\varepsilon \mathcal{D}_\varepsilon^{(2)} v_\varepsilon \right)(s, y) dy ds \right|_{C^\gamma} \\
&\lesssim \|\varphi_\varepsilon^{(1)} - \varphi\|_{C_{T, \frac{\alpha}{2}}^\alpha} \int_0^t (t-s)^{-\frac{1}{2}(\gamma-\alpha)} s^{-\frac{\alpha}{2}} |v_\varepsilon DG(u_\varepsilon) v_\varepsilon(s)|_{C^\alpha} ds \\
&\quad + \|\varphi_\varepsilon^{(2)}\|_{H_T^{-\alpha}} \int_0^t \left(|v_\varepsilon DG(u_\varepsilon) v_\varepsilon(s)|_{C^\alpha} |p_{t-s}^\varepsilon|_{C^\alpha} \right. \\
&\quad \quad \quad + |v_\varepsilon DG(u_\varepsilon) v_\varepsilon(s)|_{C^\alpha} |p_{t-s}^\varepsilon|_{C^\gamma}^{1-\alpha} |p_{t-s}^\varepsilon|_{C^{1+\gamma}}^\alpha \\
&\quad \quad \quad \left. + |v_\varepsilon DG(u_\varepsilon) v_\varepsilon(s)|_{C^\alpha} |p_{t-s}^\varepsilon|_\gamma \right) ds .
\end{aligned}$$

Since $|p_s^\varepsilon|_{C^\alpha} \lesssim s^{-\frac{1+\alpha}{2}}$ by Lemma 4.9, and

$$|v_\varepsilon DG(u_\varepsilon) v_\varepsilon(s)|_{C^\alpha} \lesssim 1$$

by assumption, we obtain

$$\begin{aligned}
|I_1|_{C^\gamma} &\lesssim t^{1-\frac{\gamma}{2}} \|\mathcal{D}_\varepsilon^{(1)}\|_{C_{T, \frac{\alpha}{2}}^\alpha} + \int_0^t (t-s)^{-\frac{1+\alpha+\gamma}{2}} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_T^{-\alpha}} ds \\
&\lesssim t^{1-\frac{\gamma}{2}} \|\mathcal{D}_\varepsilon^{(1)}\|_{C_{T, \frac{\alpha}{2}}^\alpha} + t^{\frac{1-\alpha-\gamma}{2}} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_T^{-\alpha}} .
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|I_2|_{C^\gamma} &\lesssim t^{1-\frac{1}{2}(\gamma-\tilde{\alpha})} \|\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon)\|_{C_T^{\tilde{\alpha}}} \\
&\lesssim t^{1-\frac{1}{2}(\gamma-\tilde{\alpha})} \left(\|u - u_\varepsilon\|_{C_T^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{C_T^{\tilde{\alpha}}} \right) ,
\end{aligned}$$

and Lemma 6.5 yields

$$\begin{aligned}
|I_3|_{C^\gamma} &\lesssim \varepsilon^{1-\kappa} t^{1-\frac{1}{2}(\gamma-\alpha+1)} \|\mathcal{F}(u, v)\|_{C_T^\alpha} \\
&\lesssim \varepsilon^{1-\kappa} t^{1-\frac{1}{2}(\gamma-\alpha+1)} .
\end{aligned}$$

Combining these bounds, we obtain the first estimate.

In order to prove the second estimate, we fix $0 \leq s < t \leq T$ and write

$$(\Upsilon_\varepsilon - \Upsilon)(t) - (\Upsilon_\varepsilon - \Upsilon)(s) =: \sum_{i=1}^2 \sum_{j=1}^3 J_{ij} ,$$

where

$$\begin{aligned}
J_{11} &= \int_0^s (S_\varepsilon(t-r) - S_\varepsilon(s-r)) (\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon))(r) dr , \\
J_{12} &= \int_0^s (S_\varepsilon(t-r) - S_\varepsilon(s-r)) (\mathcal{F} - \mathcal{F}_\varepsilon)(u_\varepsilon, v_\varepsilon)(r) dr , \\
J_{13} &= \int_0^s \left((S - S_\varepsilon)(t-r) - (S - S_\varepsilon)(s-r) \right) \mathcal{F}(u, v)(r) dr ,
\end{aligned}$$

and

$$J_{21} = \int_s^t S_\varepsilon(t-r) (\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon))(r) dr ,$$

$$J_{22} = \int_s^t S_\varepsilon(t-r) (\mathcal{F} - \mathcal{F}_\varepsilon)(u_\varepsilon, v_\varepsilon)(r) dr ,$$

$$J_{23} = \int_s^t (S - S_\varepsilon)(t-r) \mathcal{F}(u, v)(r) dr .$$

Using the argument from (4.12) we obtain

$$\begin{aligned} |J_{11}|_C &\lesssim |t-s|^{1-\kappa} \|\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon)\|_{\mathcal{C}_t^{\tilde{\alpha}}} \\ &\lesssim |t-s|^{1-\kappa} \left(\|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \right) \\ &\lesssim t^{1-\frac{\gamma}{2}-\kappa} |t-s|^{\frac{\gamma}{2}} \left(\|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \right) . \end{aligned}$$

Furthermore,

$$\begin{aligned} |J_{12}|_C &\lesssim \int_0^s \left| (S_\varepsilon(t-r) - S_\varepsilon(s-r)) (v_\varepsilon DG(u_\varepsilon) v_\varepsilon \mathcal{D}_\varepsilon^{(1)})(r) \right|_C dr \\ &\quad + \int_0^s \left| \int_{-\pi}^\pi (p_{t-r}^\varepsilon - p_{s-r}^\varepsilon)(\cdot - y) (v_\varepsilon DG(u_\varepsilon) v_\varepsilon \mathcal{D}_\varepsilon^{(2)})(r, y) dy \right|_C dr \\ &=: J_{12}^{(1)} + J_{12}^{(2)} . \end{aligned}$$

Arguing again as in (4.12) we obtain

$$J_{12}^{(1)} \lesssim \varepsilon^{1-\frac{\gamma}{2}-\kappa} s^\alpha |t-s|^{\frac{\gamma}{2}} \|\mathcal{D}_\varepsilon^{(1)}\|_{\mathcal{C}_{t, \frac{\alpha}{2}}^\alpha} .$$

Using Lemma 4.8 and Lemma 4.9 we obtain

$$\begin{aligned} J_{12}^{(2)} &\lesssim \int_0^s |p_{t-r}^\varepsilon - p_{s-r}^\varepsilon|_C^{1-\alpha} |p_{t-r}^\varepsilon - p_{s-r}^\varepsilon|_C^\alpha \\ &\quad \times |\mathcal{D}_\varepsilon^{(2)}(r)|_{H^{-\alpha}} |(v_\varepsilon DG(u_\varepsilon) v_\varepsilon)(r)|_{C^\alpha} dr \\ &\lesssim \int_0^s |p_{t-r}^\varepsilon - p_{s-r}^\varepsilon|_C^{1-\alpha} |p_{t-r}^\varepsilon - p_{s-r}^\varepsilon|_C^\alpha \|\mathcal{D}_\varepsilon^{(2)}\|_{H_t^{-\alpha}} dr \\ &\lesssim \int_0^s (t-s)^{\frac{\gamma}{2}} (s-r)^{-\frac{1}{2}(1+\alpha+\gamma)+\kappa} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_t^{-\alpha}} dr \\ &\lesssim s^{\frac{1}{2}(1-\alpha-\gamma)} (t-s)^{\frac{\gamma}{2}} \|\mathcal{D}_\varepsilon^{(2)}\|_{H_t^{-\alpha}} . \end{aligned}$$

Arguing as in (4.13) we obtain

$$\begin{aligned} |J_{13}|_{C_0} &\leq \varepsilon^{1-\frac{\gamma}{2}-\kappa} (t-s)^{\frac{\gamma}{2}-\kappa} \|\mathcal{F}(u, v)\|_{\mathcal{C}_t^\alpha} \\ &\lesssim \varepsilon^{1-\frac{\gamma}{2}-\kappa} (t-s)^{\frac{\gamma}{2}} , \end{aligned}$$

Moreover,

$$\begin{aligned} |J_{21}|_C &\lesssim |t-s| \|\mathcal{F}(u, v) - \mathcal{F}(u_\varepsilon, v_\varepsilon)\|_{\mathcal{C}_t^{\tilde{\alpha}}} \\ &\lesssim |t-s| \left(\|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \right) \\ &\lesssim t^{1-\frac{\gamma}{2}} |t-s|^{\frac{\gamma}{2}} \left(\|u - u_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} + \|v - v_\varepsilon\|_{\mathcal{C}_t^{\tilde{\alpha}}} \right) . \end{aligned}$$

Furthermore,

$$|J_{22}|_C \lesssim \int_s^t \left| S_\varepsilon(t-r) \left(DG(u_\varepsilon) v_\varepsilon \mathcal{D}_\varepsilon^{(1)} v_\varepsilon \right)(r, y) \right|_C dr$$

$$\begin{aligned}
& + \int_s^t \left| \int_{-\pi}^{\pi} p_{t-r}^{\varepsilon}(\cdot - y) \left(DG(u_{\varepsilon}) v_{\varepsilon} \mathcal{D}_{\varepsilon}^{(2)} v_{\varepsilon} \right)(r, y) dy \right|_{\mathcal{C}} dr \\
& =: J_{22}^{(1)} + J_{22}^{(2)}.
\end{aligned}$$

The first term can easily be estimated by

$$\begin{aligned}
J_{22}^{(1)} & \lesssim \int_s^t r^{-\frac{\alpha}{2}} \|\mathcal{D}_{\varepsilon}^{(1)}\|_{C_{t, \frac{\alpha}{2}}^{\alpha}} dr \lesssim (t-s)^{1-\frac{\alpha}{2}-\kappa} \|\mathcal{D}_{\varepsilon}^{(1)}\|_{C_{t, \frac{\alpha}{2}}^{\alpha}} \\
& \lesssim (t-s)^{\frac{\gamma}{2}} t^{1-\frac{\alpha}{2}-\frac{\gamma}{2}-\kappa} \|\mathcal{D}_{\varepsilon}^{(1)}\|_{C_{t, \frac{\alpha}{2}}^{\alpha}}.
\end{aligned}$$

Lemma 4.8 and Lemma 4.9 yield

$$\begin{aligned}
J_{22}^{(2)} & \lesssim \int_s^t |p_{t-r}^{\varepsilon}|_{\mathcal{C}}^{1-\alpha} |p_{t-r}^{\varepsilon}|_{\mathcal{C}^1}^{\alpha} |(v_{\varepsilon} DG(u_{\varepsilon}) v_{\varepsilon})(r)|_{\mathcal{C}^{\alpha}} |\mathcal{D}_{\varepsilon}^{(2)}(r)|_{H^{-\alpha}} dr \\
& \lesssim \int_s^t (t-r)^{-\frac{1}{2}(1+\alpha)} \|\mathcal{D}_{\varepsilon}^{(2)}\|_{H_t^{-\alpha}} dr \lesssim (t-s)^{\frac{1}{2}(1-\alpha)} \|\mathcal{D}_{\varepsilon}^{(2)}\|_{H_t^{-\alpha}} \\
& \lesssim t^{\frac{1}{2}(1-\alpha-\gamma)} (t-s)^{\frac{\gamma}{2}} \|\mathcal{D}_{\varepsilon}^{(2)}\|_{H_t^{-\alpha}}.
\end{aligned}$$

Finally, by the argument in (4.14),

$$|J_{23}|_{\mathcal{C}_0} \lesssim \varepsilon^{1-\kappa} (t-s)^{\frac{1}{2}(1+\alpha)} \|\mathcal{F}(u, v)\|_{\mathcal{C}_t^{\alpha}} \lesssim \varepsilon^{1-\kappa} (t-s)^{\frac{1}{2}(1+\alpha)}.$$

Putting everything together, we obtain the desired bound. \square

The following lemma has been used in the proof above.

Lemma 4.8. *Let $0 \leq \alpha, \gamma \leq 1$. For $p \in \mathcal{C}^{1+\gamma}$, $\varphi \in \mathcal{C}^{\alpha}$, and $\psi \in H^{-\alpha}$ we have*

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} p(\cdot - y) \varphi(y) \psi(y) dy \right|_{\mathcal{C}^{\gamma}} & \lesssim |\psi|_{H^{-\alpha}} \left(|p|_{\mathcal{C}^{\alpha}} |\varphi|_{\mathcal{C}^{\alpha}} + |p|_{\gamma} |\varphi|_{\mathcal{C}^{\alpha}} \right. \\
& \quad \left. + |p|_{\mathcal{C}^{\gamma}}^{1-\alpha} |p|_{\mathcal{C}^{1+\gamma}}^{\alpha} |\varphi|_{\mathcal{C}} \right).
\end{aligned}$$

Proof. First we fix $x \in [-\pi, \pi]$ to obtain

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} p(x-y) \varphi(y) \psi(y) dy \right| & \leq |p(x-\cdot) \varphi(\cdot)|_{\mathcal{C}^{\alpha}} |\psi|_{H^{-\alpha}} \\
& \lesssim |p|_{\mathcal{C}^{\alpha}} |\varphi|_{\mathcal{C}^{\alpha}} |\psi|_{H^{-\alpha}}.
\end{aligned}$$

Next we fix $x, y \in [-\pi, \pi]$, and set

$$f_{x,y}(z) := \frac{p(x-z) - p(y-z)}{|x-y|^{\gamma}}.$$

It follows that

$$\left| \int_{-\pi}^{\pi} \frac{p(x-z) - p(y-z)}{|x-y|^{\gamma}} \varphi(z) \psi(z) dz \right| \leq |f_{x,y} \varphi|_{\mathcal{C}^{\alpha}} |\psi|_{H^{-\alpha}}.$$

Since $|f_{x,y}|_{\mathcal{C}} \leq |p|_{\gamma}$, and

$$|f_{x,y}|_{\mathcal{C}^{\alpha}} \leq |f_{x,y}|_{\mathcal{C}}^{1-\alpha} |f_{x,y}|_{\mathcal{C}^1}^{\alpha} \leq |p|_{\mathcal{C}^{\gamma}}^{1-\alpha} |p|_{\mathcal{C}^{1+\gamma}}^{\alpha},$$

we infer that

$$\begin{aligned}
& \left| \int_{-\pi}^{\pi} \frac{p(x-z) - p(y-z)}{|x-y|^{\gamma}} \varphi(z) \psi(z) dz \right| \\
& \lesssim \left(|p|_{\gamma} |\varphi|_{\mathcal{C}^{\alpha}} + |p|_{\mathcal{C}^{\gamma}}^{1-\alpha} |p|_{\mathcal{C}^{1+\gamma}}^{\alpha} |\varphi|_{\mathcal{C}} \right) |\psi|_{H^{-\alpha}}.
\end{aligned}$$

Taking the supremum over x (resp. x, y), we obtain the desired estimate. \square

We shall apply this result to the approximating heat kernel $p = p_t^\varepsilon$. The following lemma contains the relevant estimates.

Lemma 4.9. *Let $\alpha \geq 0$ and $\delta \in [0, 1]$. For all $0 \leq s < t \leq T$, $\varepsilon \in (0, 1]$, and $\kappa \in (0, \frac{1}{2})$ we have*

$$\begin{aligned} |p_t^\varepsilon|_\alpha &\lesssim t^{-\frac{1}{2}(1+\alpha)}, \\ |p_t^\varepsilon - p_s^\varepsilon|_\alpha &\lesssim s^{-\frac{1}{2}(1+\alpha)-\delta+\kappa}(t-s)^\delta. \end{aligned}$$

Proof. Since $p_t^\varepsilon(x) = \sum_k e^{-f(\varepsilon k)k^2 t} e^{ikx}$, we obtain for $n = 0, 1, \dots$,

$$|p_t^\varepsilon|_n \leq \sum_{k \in \mathbb{Z}} |k|^n e^{-f(\varepsilon k)k^2 t} \leq \sum_{k \in \mathbb{Z}} |k|^n e^{-2c_f k^2 t} \lesssim t^{-\frac{1}{2}(n+1)}.$$

Similarly, setting $p = \frac{1}{2}(n+1) + \delta - \kappa$ and using that $f \geq 2c_f$, we obtain

$$\begin{aligned} |p_t^\varepsilon - p_s^\varepsilon|_n &\leq \sum_{k \in \mathbb{Z}} |k|^n (e^{-f(\varepsilon k)k^2 t} - e^{-f(\varepsilon k)k^2 s}) \\ &\leq \sum_{k \in \mathbb{Z}} |k|^n e^{-sk^2 f(\varepsilon k)} |1 - e^{-(t-s)k^2 f(\varepsilon k)}| \\ &\lesssim \sum_{k \in \mathbb{Z}} |k|^n \cdot s^{-p} k^{-2p} f(\varepsilon k)^{-p} \cdot (t-s)^\delta k^{2\delta} f(\varepsilon k)^\delta \\ &\lesssim s^{-p} (t-s)^\delta \sum_{k \in \mathbb{Z}} |k|^{-1+2\kappa} \lesssim s^{-p} (t-s)^\delta. \end{aligned}$$

The result for $\alpha \geq 0$ follows now by interpolation. \square

5. ROUGH PATH ESTIMATES

In this section we treat the stability of approximations of the term involving $G(u) \partial_x u$. Throughout the calculations we will make heavy use of the rough path bounds provided in Appendix A. We will fix deterministic data $(u, u_\varepsilon, X, \text{etc.})$ and derive bounds based on the regularity of these data. There will be no randomness involved.

We fix Hölder exponents $\tilde{\alpha} \leq \alpha \in (1/3, 1/2)$. We will often use a number $\kappa > 0$. This value need not coincide with the value of κ fixed in Section 2. We also fix rough path valued mappings $(X(t), \mathbf{X}(t))$ and $(X_\varepsilon(t), \mathbf{X}_\varepsilon(t))$. To be more precise, we will assume that the mappings $[0, T] \ni t \mapsto X \in \mathcal{C}^\alpha$ and $[0, T] \ni t \mapsto \mathbf{X} \in \mathcal{B}^{2\alpha}$ are continuous and that for every t the functions $x \mapsto X(t, x)$ and $(x, y) \mapsto \mathbf{X}(t; x, y)$ satisfy the consistency relation (A.3). The functions $(X_\varepsilon(t), \mathbf{X}_\varepsilon(t))$ will be assumed to satisfy the same conditions.

We will also fix functions $u, u_\varepsilon \in \mathcal{C}_T^\alpha$. We assume that for every t the function u is controlled by X . More precisely, we will assume that for every $t \in [0, T]$ there are $u'(t)$ and $R_u(t)$

$$t \mapsto u(t) \in \mathcal{C}^\alpha \quad t \mapsto u'(t) \in \mathcal{C}^\alpha \quad t \mapsto R_u \in \mathcal{B}^{2\alpha}$$

are continuous and that for every t the maps

$$x \mapsto u(t, x), X(t, x) \quad \text{and} \quad (x, y) \mapsto R_u(t; x, y), \mathbf{X}(t, x, y) \quad (5.1)$$

satisfy the relation (A.4).

In the same way we will assume that the u_ε are controlled by X_ε for $t > \varepsilon^2$. We will assume that for $t \in (\varepsilon^2, T]$ there are functions u'_ε and R_{u_ε} that are continuous in time so that for every t the $u_\varepsilon, u'_\varepsilon, R_{u_\varepsilon}, \mathbf{X}_\varepsilon$ satisfy the relation (A.4).

Throughout this section we will make the standing assumption that the norms

$$\|X\|_{\mathcal{C}_T^\alpha}, \|X_\varepsilon\|_{\mathcal{C}_T^\alpha}, \|\mathbf{X}\|_{\mathcal{B}_T^{2\alpha}}, \|\mathbf{X}_\varepsilon\|_{\mathcal{B}_T^{2\alpha}}, \|u\|_{\mathcal{C}_T^\alpha}, \|u_\varepsilon\|_{\mathcal{C}_T^\alpha}, \|u'\|_{\mathcal{C}_T^\alpha}, \|u'_\varepsilon\|_{\mathcal{C}_{[\varepsilon^2, T]}^\alpha}$$

are bounded by a large constant K . We will also assume that for $s > 0$

$$|R_u(s)|_{2\alpha} \leq K s^{-\frac{\beta}{2}}, \quad |R_u(s)|_{2\tilde{\alpha}} \leq K s^{-\frac{\tilde{\beta}}{2}}, \quad (5.2)$$

and for $s > \varepsilon^2$

$$|R_{u_\varepsilon}(s)|_{2\alpha} \leq K(s - \varepsilon^2)^{-\frac{\beta}{2}}, \quad |R_{u_\varepsilon}(s)|_{2\tilde{\alpha}} \leq K(s - \varepsilon^2)^{-\frac{\tilde{\beta}}{2}} \quad (5.3)$$

for some $\tilde{\beta} \leq \beta \in (0, \frac{1}{2})$.

Most of the constants that appear in this section (or that are suppressed when we write \lesssim) depend on the choice of this constant K .

The main objective of this section is to study the behaviour of the following term for small ε :

$$\begin{aligned} \Xi_\varepsilon^i(t, \cdot) &= \int_0^t S_\varepsilon(t-s) \left[G(u_\varepsilon(s))_j^i D_\varepsilon u_\varepsilon(s)^j \right] ds \\ &+ \int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) \partial_k G(u_\varepsilon(s, y))_j^i u'_\varepsilon(s, y)_l^k D_\varepsilon \mathbf{X}_\varepsilon(s; y)^{l,m} u'_\varepsilon(s, y)_m^j dy ds. \end{aligned} \quad (5.4)$$

Here, as above, we have used the notation

$$D_\varepsilon \mathbf{X}_\varepsilon(s; y) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \mathbf{X}_\varepsilon(s; y, y + \varepsilon z) \mu(dz).$$

Note that in (5.4) we have included indices to capture the trilinear structure in the second term on the right-hand side. The linear algebra does not play a crucial role for our argument, and for simplicity we will omit the indices for most of the argument.

Denote by $\Xi(t, x)$ the function

$$\Xi(t, x) = \int_0^t \left[\int_{-\pi}^\pi p_{t-s}(x - y) G(u(s, y)) dy u(s, y) \right] ds. \quad (5.5)$$

Throughout this section we will make the additional assumption that the function G is bounded with bounded derivatives up to order three. This assumption is removed in Section 2 using an appropriate stopping time.

As explained in Section 2, we have included the extra term

$$\int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(\cdot - y) \partial_k G(u_\varepsilon(s, y))_j^i u'_\varepsilon(s, y)_l^k D_\varepsilon \mathbf{X}_\varepsilon(s; y)^{l,m} u'_\varepsilon(s, y)_m^j dy ds$$

on the right-hand side of (5.4) to ensure that the Ξ_ε approximates Ξ . As discussed in Sections 2 and 4, this term gives rise to the extra term in the limit.

The main goal of this section is to establish the following bounds.

Proposition 5.1. *For every $\gamma \in (0, 1)$ and for any $\kappa > 0$ small enough and $\tilde{\gamma} \in (\gamma, 1)$ we have for any $t \in [0, T]$*

$$\begin{aligned} |\Xi(t) - \Xi_\varepsilon(t)|_{\mathcal{C}^\gamma} &\lesssim \mathcal{D}_\varepsilon t^{\frac{1+\tilde{\alpha}-\gamma-\tilde{\beta}-\kappa}{2}} \\ &+ |\varepsilon|^{3\alpha-1} t^{1-\frac{2\alpha+\gamma+\beta+\kappa}{2}} + |\varepsilon|^{(1+\tilde{\alpha}-\tilde{\beta}-\gamma-\kappa)\wedge 1} + \varepsilon^{(\tilde{\gamma}-\gamma-\kappa)} t^{1-\tilde{\gamma}+\alpha-\beta}. \end{aligned} \quad (5.6)$$

For any $\gamma < \frac{1}{2}$ we get

$$\|\Xi(t) - \Xi_\varepsilon(t)\|_{C^{\frac{\gamma}{2}}([\varepsilon^2, T], \mathcal{C})} \lesssim \mathcal{D}_\varepsilon T^{\frac{1+\tilde{\alpha}-\tilde{\beta}-\gamma-\kappa}{2}} + \varepsilon^{3\alpha-1} + \varepsilon^{(2-2\tilde{\alpha}-\tilde{\beta}-\gamma)\wedge 1}. \quad (5.7)$$

Here we use \mathcal{D}_ε as abbreviation for

$$\begin{aligned} \mathcal{D}_\varepsilon = & \left[\|X - X_\varepsilon\|_{C_T^{\tilde{\alpha}}} + \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}_T^{2\tilde{\alpha}}} + \|u - u_\varepsilon\|_{C_T^{\tilde{\alpha}}} + \|u' - u'_\varepsilon\|_{C_{[\varepsilon^2, T]}^{\tilde{\alpha}}} \right. \\ & \left. + \|R_u - R_{u_\varepsilon}\|_{\mathcal{B}_{[\varepsilon^2, T], \tilde{\beta}}^{2\tilde{\alpha}}} \right]. \end{aligned}$$

The constant which is suppressed in the notation depends on T and K .

In the proof of Proposition 5.1 we will need to understand the behaviour of rescaled rough integrals under approximations. This is the content of the following lemma. It is a modification of the scaling lemma [HW10, Lemma 5.1], which is also cited in the appendix as Lemma A.5.

Lemma 5.2. *Let (X, \mathbf{X}) be an α -rough path and let Y, Z be controlled by X . Furthermore, assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function for which*

$$|f|_{1,1} = \sum_{k \in \mathbb{Z}} \sup_{x \in [k, k+1]} |f(x)| + |f'(x)|$$

is finite. For a fixed step width $\varepsilon > 0$ we set $x_k = \varepsilon k$ and $N_\varepsilon = \lfloor \frac{\pi}{\varepsilon} \rfloor$. For any $\lambda > 0$ we consider the approximating rough integrals

$$\begin{aligned} I_+^{\lambda, \varepsilon} = & \sum_{k=-N_\varepsilon}^{N_\varepsilon-1} \left[f(\lambda x_k) Y(x_k) \left(Z(x_{k+1}) - Z(x_k) \right) \right. \\ & \left. + f(\lambda x_k) Y'(x_k) \mathbf{X}(x_k, x_{k+1}) Z'(x_k) \right], \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} I_-^{\lambda, \varepsilon} = & \sum_{k=-N_\varepsilon}^{N_\varepsilon-1} \left[f(\lambda x_{k+1}) Y(x_{k+1}) \left(Z(x_k) - Z(x_{k+1}) \right) \right. \\ & \left. + f(\lambda x_{k+1}) Y'(x_{k+1}) \mathbf{X}(x_{k+1}, x_k) Z'(x_{k+1}) \right], \end{aligned} \quad (5.9)$$

Then, as long as $\varepsilon\lambda < 1$, we have the bound

$$\left| \int_{-\pi}^{\pi} f(\lambda x) Y(x) dZ(x) \mp I_{\pm}^{\lambda, \varepsilon} \right| \lesssim \varepsilon^{3\alpha-1} \lambda^{2\alpha-1} |f|_{1,1} \Lambda(X, Y, Z), \quad (5.10)$$

where

$$\begin{aligned} \Lambda(X, Y, Z) := & |Y|_0 |Z|_\alpha + |R_Y|_{2\alpha} |Z|_\alpha + |X|_\alpha |Y'|_{\mathcal{C}} |R_Z|_{2\alpha} \\ & + |\mathbf{X}|_{2\alpha} \left(|Y'|_0 |Z'|_{\mathcal{C}^\alpha} + |Y'|_{\mathcal{C}^\alpha} |Z'|_0 \right) + |X|_\alpha^2 |Y'|_{\mathcal{C}} |Z|_\alpha. \end{aligned}$$

Remark 5.3. The trilinear expression $Y' \mathbf{X} Z'$ in (5.8) and (5.9) should be interpreted as above in (5.4). We omit the indices to keep the notation short.

Remark 5.4. The term $|f|_{C^{2\alpha}} \Lambda(X, Y, Z)$ arises when evaluating the expressions on the right-hand side of (A.6) and (A.7) for the rough paths fY and Z . We will use the fact that there is no term in $\Lambda(X, Y, Z)$ that includes the product $|R_Y|_{2\alpha} |R_Z|_{2\alpha}$.

This will be important for our application, as for the rough paths that we work with the remainder will not be bounded uniformly in time but only by $s^{-\frac{\beta}{2}}$.

Proof. Let us start by treating the approximation $I_+^{\lambda,\varepsilon}$. One can decompose the expression on the left-hand side of (5.10) to obtain

$$\left| \int_{-\pi}^{\pi} f(\lambda x) Y(x) dZ(x) - I_+^{\lambda,\varepsilon} \right| \leq \sum_{k=-N_\varepsilon}^{N_\varepsilon-1} |I_k^\lambda - I_k^{\lambda,\varepsilon}| + |E|. \quad (5.11)$$

Here

$$I_k^\lambda = \int_{y_k}^{y_{k+1}} f(y) Y_\lambda(y) dZ_\lambda(y)$$

is a rescaled version of the rough integral. We have used the following notation. We set $y_k = \lambda x_k = \lambda \varepsilon k$ and the paths $X_\lambda, Y_\lambda, Z_\lambda$ are rescaled versions of X, Y, Z . In particular, note that $Y_\lambda(y) = Y(y/\lambda)$ is a controlled rough path with respect to $(X_\lambda(y), \mathbf{X}_\lambda(y, \hat{y})) = (X(y/\lambda), \mathbf{X}(y/\lambda, \hat{y}/\lambda))$ with derivative process given by $Y'_\lambda(y) = Y'(y/\lambda)$, and similarly for Z_λ .

The integral approximation term on the right-hand side of (5.11) is defined by

$$I_k^{\lambda,\varepsilon} = f(y_k) Y_\lambda(y_k) (Z_\lambda(y_{k+1}) - Z_\lambda(y_k)) \\ + f(y_k) Y'_\lambda(y_k) \mathbf{X}_\lambda(y_k, y_{k+1}) Z'_\lambda(y_k)$$

and the error term is given by

$$E = \int_{\lambda \varepsilon N_\varepsilon}^{\lambda \pi} f(z) Y_\lambda(z) dZ_\lambda(z) + \int_{-\lambda \pi}^{-\lambda \varepsilon N_\varepsilon} f(z) Y_\lambda(z) dZ_\lambda(z). \quad (5.12)$$

Using (A.6) and (A.7), this expression can directly be bounded

$$|E| \lesssim (\varepsilon \lambda)^\alpha \lambda^{-\alpha} |f|_{1,1} \Lambda(X, Y, Z).$$

Due to the assumption that $\lambda \varepsilon < 1$ and $\alpha < \frac{1}{2}$ we have

$$(\varepsilon \lambda)^\alpha \leq (\varepsilon \lambda)^{3\alpha-1},$$

which gives the right scaling.

For the other terms on the right-hand side of (5.11) we get

$$|I_k^\lambda - I_k^{\lambda,\varepsilon}| \lesssim \sup_{x \in [\lambda k \varepsilon, \lambda(k+1)\varepsilon]} (|f(x)| + |f'(x)|) \\ (\varepsilon \lambda)^{3\alpha} \lambda^{-\alpha} \Lambda(X, Y, Z). \quad (5.13)$$

Now we sum this bound over k and obtain

$$\sum_{k=-N_\varepsilon}^{N_\varepsilon} |I_k^\lambda - I_k^{\lambda,\varepsilon}| \lesssim \sum_{l \in \mathbb{Z}} M_l \alpha_l (\varepsilon \lambda)^{3\alpha} \lambda^{-\alpha} \Lambda(X, Y, Z), \quad (5.14)$$

where $\alpha_l = \sup_{x \in [l, l+1]} (|f(x)| + |f'(x)|)$ and

$$M_l = \#\{k \in \mathbb{Z}: [\lambda k \varepsilon, \lambda(k+1)\varepsilon] \cap [l, l+1] \neq \emptyset\}.$$

We get

$$M_l \leq (\varepsilon \lambda)^{-1} + 2 \leq 3(\varepsilon \lambda)^{-1}$$

due to the assumption $\lambda \varepsilon < 1$. Plugging this into (5.10) and recalling the definition of $|f|_{1,1}$ finishes the proof.

The proof for $I_-^{\lambda,\varepsilon}$ is similar. In order to apply the same argument, one only has to check that the bound corresponding to (5.13) also holds in the present context. To this end define $I_{k,-}^{\lambda,\varepsilon}$ to be the k -th summand in the approximation (5.9) and calculate

$$\begin{aligned} I_{k,-}^{\lambda,\varepsilon} + I_k^{\lambda,\varepsilon} = & \left(f(y_k) Y(y_k) - f(y_{k+1}) Y_\lambda(y_{k+1}) \right) \left(Z_\lambda(y_{k+1}) - Z_\lambda(y_k) \right) \\ & + \left(f(y_k) Y'_\lambda(y_k) \mathbf{X}_\lambda(y_k, y_{k+1}) Z'_\lambda(y_k) \right. \\ & \quad \left. - f(y_{k+1}) Y'_\lambda(y_{k+1}) \mathbf{X}_\lambda(y_k, y_{k+1}) Z'_\lambda(y_{k+1}) \right) \\ & + f(y_{k+1}) Y'_\lambda(y_{k+1}) \delta X_\lambda(y_k, y_{k+1}) \delta X_\lambda(y_k, y_{k+1}) Z'_\lambda(y_{k+1}), \end{aligned} \quad (5.15)$$

Here we have used the consistency relation (A.3) for \mathbf{X} . The term in the last line of (5.15) can be expressed as

$$\begin{aligned} f(\lambda y_{k+1}) \left[\left(\delta Y(y_{k+1}, y_k) + R_Y(y_{k+1}, y_k) \right) \delta Z(y_{k+1}, y_k) \right. \\ \left. + Y'(y_{k+1}) \delta X(x_{k+1}, x_k) R_Z(y_{k+1}, y_k) \right]. \end{aligned}$$

So using the smoothness of f one can see that these terms cancel with the terms in the first line up to an error of order $(\varepsilon\lambda)^{3\alpha}$. The term in the second line of (5.15) is also of order $(\varepsilon\lambda)^{3\alpha}$. Therefore, we finally get

$$|I_{k,-}^{\lambda,\varepsilon} + I_k^{\lambda,\varepsilon}| \lesssim (\varepsilon\lambda)^{3\alpha} \lambda^{-\alpha} \Lambda(X, Y, Z).$$

Then the rest of the proof is the same as before. \square

We now have all the ingredients to proceed to the proof of Proposition 5.1. We prove the bounds (5.6) on the space regularity and (5.7) on the time regularity separately.

Proof of (5.6). First we shall reduce the argument to the case where D_ε is the simple finite difference operator

$$D_\varepsilon^z u(x) = \frac{1}{\varepsilon z} (u(x + \varepsilon z) - u(x)).$$

To this end we exchange the order of integration in (5.4). Recalling the definition of the operator D_ε in Assumption 1.3 we can write

$$\Xi_\varepsilon(t, x) = \int_{\mathbb{R}} z \Xi_\varepsilon^z(t, x) \mu(dz),$$

where

$$\begin{aligned} \Xi_\varepsilon^z(t, x) = & \int_0^t \int_{-\pi}^\pi p_{t-s}^\varepsilon(x - y) \left[G(u_\varepsilon(s, y)) D_\varepsilon^z u_\varepsilon(s, y) \right. \\ & \left. + DG(u_\varepsilon(s, y)) u'_\varepsilon(s, y) D_\varepsilon^z \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) \right] dy ds. \end{aligned} \quad (5.16)$$

For convenience, here and in the sequel we have dropped all the indices.

Recalling that $\int_{\mathbb{R}} z \mu(dz) = 1$ according to Assumption 1.3, we see that

$$\Xi(t, x) - \Xi_\varepsilon(t, x) = \int_{\mathbb{R}} (\Xi(t, x) - \Xi_\varepsilon^z(t, x)) z \mu(dz). \quad (5.17)$$

Therefore, most of the proof will be concerned with deriving z -dependent bounds on $\Xi^z(t, x) - \Xi_\varepsilon^z(t, x)$. Only in the last step we will integrate over z .

Recall the definition (3.45) of $A_\varepsilon(t)$ and $A(t)$. It will be convenient to factorise the operators $S(t)$ and $S_\varepsilon(t)$ as

$$S(t) = A(t) S(c_f t) \quad \text{and} \quad S_\varepsilon(t) = A_\varepsilon(t) S(c_f t). \quad (5.18)$$

We write the semigroups in this form, because it is not straightforward to check that S_ε has the right regularising properties in Hölder spaces. The factorisation (5.18) allows us to use the known scaling property of the heat kernels $p_{c_f(t-s)}$ to apply Lemma 5.2 and to obtain a smoothing effect. Then we will use Lemma 6.4 to conclude that $A_\varepsilon(t-s)$ preserves (most of) the regularity.

For every z one can write

$$\Xi(t, x) - \Xi_\varepsilon^z(t, x) = I_1^z(t, x) + I_2(t, x), \quad (5.19)$$

where I_1^z and I_2 are given by

$$\begin{aligned} I_1^z(t) &= \int_0^t A_\varepsilon(t-s) \left(\int_{-\pi}^\pi p_{c_f(t-s)}(\cdot - y) G(u(s, y)) dy u(s, y) \right. \\ &\quad - \int_{-\pi}^\pi p_{c_f(t-s)}(\cdot - y) \left[G(u_\varepsilon(s, y)) D_\varepsilon^z(u_\varepsilon(s, y)) \right. \\ &\quad \left. \left. + DG(u_\varepsilon(s, y)) u'_\varepsilon(s, y) D_\varepsilon^z \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) \right] dy \right) ds, \\ I_2(t) &= \int_0^t \left[A(t-s) - A_\varepsilon(t-s) \right] \int_{-\pi}^\pi p_{c_f(t-s)}(\cdot - y) G(u(s, y)) dy u(s, y) ds. \end{aligned}$$

We divide the bounds on the terms I_1^z, I_2 into two lemmas and start with I_1^z .

Lemma 5.5. *For $\gamma \in (0, 1)$ and $\kappa > 0$ small enough we get for any $t \in [0, T]$*

$$\begin{aligned} |I_1^z(t)|_{\mathcal{C}^\gamma} &\lesssim \mathcal{D}_\varepsilon t^{\frac{1+\tilde{\alpha}-\gamma-\tilde{\beta}-\kappa}{2}} + |\varepsilon z|^{3\alpha-1} t^{1-\frac{2\alpha+\gamma+\beta+\kappa}{2}} \\ &\quad + |\varepsilon z|^{(1+\tilde{\alpha}-\tilde{\beta}-\gamma-\kappa)\wedge 1}. \end{aligned} \quad (5.20)$$

Proof of Lemma 5.5. Throughout the calculations we will often drop the explicit dependence of u_ε and \mathbf{X}_ε on the time variable. We introduce the quantity

$$\begin{aligned} J^z(s, t; x) &:= \int_{-\pi}^\pi p_{c_f(t-s)}(x-y) G(u(s, y)) dy u(s, y) \\ &\quad - \int_{-\pi}^\pi p_{c_f(t-s)}(x-y) \left[G(u_\varepsilon(s, y)) D_\varepsilon^z u_\varepsilon(s, y) \right. \\ &\quad \left. + DG(u_\varepsilon(s, y)) u'_\varepsilon(s, y) D_\varepsilon^z \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) \right] dy \\ &=: J_0^z(s, t; x) - J_\varepsilon^z(s, t; x). \end{aligned} \quad (5.21)$$

Then we get

$$I_1^z(t, \cdot) = \int_0^t A_\varepsilon(t-s) J^z(s, t; \cdot) ds. \quad (5.22)$$

In order to get bounds on the spatial \mathcal{C}^γ norm of I_1^z we will first derive bounds on the $\mathcal{C}^{\gamma+\kappa}$ norm of the $J^z(s, t)$ for some $\kappa > 0$. Then we will use Lemma 6.4 to conclude that the operator $A(t-s)$ is a contraction from $\mathcal{C}^{\gamma+\kappa}$ to \mathcal{C}^γ . In order to get these bounds on J^z we will actually give a bound on the \mathcal{C}^1 norm and on the \mathcal{C} norm and then interpolate between these.

Differentiating under the integral yields

$$\begin{aligned} \partial_x J^z(s, t; x) &= \int_{-\pi}^{\pi} \partial_x p_{cf}(t-s)(x-y) G(u(s, y)) dy u(s, y) \\ &\quad - \int_{-\pi}^{\pi} \partial_x p_{cf}(t-s)(x-y) \left[G(u_\varepsilon(s, y)) D_\varepsilon^z u_\varepsilon(s, y) \right. \\ &\quad \left. + DG(u_\varepsilon(s, y)) u'_\varepsilon(s, y) D_\varepsilon^z \mathbf{X}_\varepsilon(s, y) u'_\varepsilon(s, y) \right] dy \\ &= \partial_x J_0^z(s, t; x) - \partial_x J_\varepsilon^z(s, t; x). \end{aligned} \quad (5.23)$$

We will need to bound $J(s, t; x)$ as well as $\partial_x J(s, t; x)$ uniformly in x . Both calculations are similar and we will present in detail only the more difficult case $\partial_x J(s, t; x)$. The calculations for $J(s, t; x)$ follow the same lines, only with some changes in the exponent of $(t-s)$. From now on we will assume without loss of generality that $x = 0$ and we will drop this argument in the notation.

First, we will establish that for every z the term $\partial_x J_\varepsilon^z(s, t)$ approximates the rough integral

$$\int_{-\pi}^{\pi} \partial_x p_{cf}(t-s)(y) G(u_\varepsilon(y)) du_\varepsilon(y). \quad (5.24)$$

This will be done in several steps. In the case where $(t-s)^{1/2}$ is larger than the discretization $\varepsilon|z|$ and when $s > \varepsilon^2$ we can apply Lemma 5.2. These calculations will be separated into the cases $z > 0$ and $z < 0$. In the case where $s \leq \varepsilon^2$ the path u_ε cannot be interpreted as a rough path. Also when $(t-s)^{1/2} < \varepsilon|z|$ Lemma 5.2 provides no information. In both cases we will give a brute force bound.

We start with the case

$$s > \varepsilon^2, \quad (t-s) > (\varepsilon z)^2, \quad \text{and} \quad z > 0. \quad (5.25)$$

We separate the integral over $[-\pi, \pi]$ into integrals over intervals of length εz . To this end we use the notation

$$N_{\varepsilon z} = \left\lfloor \frac{\pi}{\varepsilon z} \right\rfloor \quad \text{and} \quad y_k = k\varepsilon z. \quad (5.26)$$

Then we can write

$$\begin{aligned} \partial_x J_\varepsilon^z(s, t) &= \frac{1}{\varepsilon z} \int_0^{\varepsilon z} \left[\sum_{k=-N_{\varepsilon z}}^{N_{\varepsilon z}-2} \partial_x p_{cf}(t-s)(y_k + y) G(u_\varepsilon(y_k + y)) \right. \\ &\quad \times \left(u_\varepsilon(y_{k+1} + y) - u_\varepsilon(y_k + y) \right) \\ &\quad + \partial_x p_{cf}(t-s)(y_k + y) DG(u_\varepsilon(y_k + y)) u'_\varepsilon(y_k + y) \\ &\quad \left. \times \mathbf{X}_\varepsilon(y_k + y, y_{k+1} + y) u'_\varepsilon(y_k + y) \right] dy \\ &\quad + E_1(z), \end{aligned} \quad (5.27)$$

where the error term E_1 is given as

$$\begin{aligned} E_1(z) &= \frac{1}{\varepsilon z} \int_{[(N_{\varepsilon z}-1)\varepsilon z, \pi] \cup [-\pi, -N_{\varepsilon z}\varepsilon z]} \left[\partial_x p_{cf}(t-s) G(u_\varepsilon) \delta_{\varepsilon z} u_\varepsilon \right. \\ &\quad \left. + \partial_x p_{cf}(t-s) DG(u_\varepsilon) u'_\varepsilon \mathbf{X}_\varepsilon u'_\varepsilon \right] dy. \end{aligned} \quad (5.28)$$

Here we have used the abbreviation $\delta_{\varepsilon z} u_\varepsilon(y) = u_\varepsilon(y + \varepsilon z) - u_\varepsilon(y)$.

On the domain of integration in (5.28) the term $\partial_x p_{c_f(t-s)}$ is uniformly bounded. Actually, if $(t-s) > 1$ this term is uniformly bounded on all of \mathbb{R} and otherwise condition (5.25) implies that on the domain of integration the argument is bounded away from zero. Therefore, we can bound

$$\begin{aligned} |E_1| &\lesssim |G|_{\mathcal{C}} |u_\varepsilon|_{3\alpha-1}(\varepsilon z)^{3\alpha-1} + |DG|_{\mathcal{C}} |u'_\varepsilon|_{\mathcal{C}}^2 |\mathbf{X}_\varepsilon|_{3\alpha-1}(\varepsilon z)^{3\alpha-1} \\ &\lesssim (\varepsilon z)^{3\alpha-1}. \end{aligned} \quad (5.29)$$

Here we have used that $3\alpha - 1 < \alpha$.

We want to show that uniformly in $y \in [0, \varepsilon z]$ the sum in square brackets on the right-hand side of (5.27) approximates the rough integral (5.24). To this end we apply Lemma 5.2 with

$$\begin{aligned} f_{(t-s)}(y) &= (t-s) \partial_x p_{c_f(t-s)}(y \sqrt{t-s}) \\ Y &= G(u_\varepsilon(y)). \end{aligned} \quad (5.30)$$

Actually, in (5.30) the function $f(y)$ is only defined for $y \in [-\pi(t-s)^{-1/2}, \pi(t-s)^{-1/2}]$ but using the reflection principle

$$p_t(y) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-(y + 2\pi k)^2}{4t}\right)$$

it can be seen easily that $f_{(t-s)}$ can be extended to all of \mathbb{R} so that

$$\sup_t \|f_t\|_{1,1} < \infty.$$

Also $Y(y) = G(u_\varepsilon(y))$ is a rough path controlled by $(X_\varepsilon, \mathbf{X}_\varepsilon)$ with rough path derivative

$$Y'(y) = DG(u_\varepsilon(y)) u'_\varepsilon(y), \quad (5.31)$$

and remainder

$$\begin{aligned} R_Y(x, y) &= DG(u_\varepsilon(x)) R_{u_\varepsilon}(x, y) \\ &+ \int_0^1 \left[DG(\lambda u_\varepsilon(y) + (1-\lambda)u_\varepsilon(x)) - DG(u_\varepsilon(x)) \right] (u_\varepsilon(y) - u_\varepsilon(x)) d\lambda. \end{aligned} \quad (5.32)$$

Recalling the boundedness assumption from the beginning of this section and in particular (5.2), we obtain

$$|Y|_{\mathcal{C}^\alpha} \lesssim 1, \quad |Y'|_{\mathcal{C}^\alpha} \lesssim 1, \quad |R_Y|_{2\alpha} \lesssim s^{-\frac{\beta}{2}}. \quad (5.33)$$

Now we are ready to apply Lemma 5.2. We can conclude that for every y in $[0, \varepsilon z]$ the sum in square brackets on the right-hand side of (5.27) can be written as

$$\int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)}(y) G(u_\varepsilon(y)) du_\varepsilon(y) + E_2(y, z), \quad (5.34)$$

where the error term $E_2(z)$ is bounded by

$$|E_2(y, z)| \lesssim (\varepsilon z)^{3\alpha-1} (t-s)^{-\frac{1-2\alpha}{2}} s^{-\frac{\beta}{2}}. \quad (5.35)$$

Here we have used the definition of $\Lambda(X, Y, Z)$ and the a priori bounds on u_ε and on $G(u_\varepsilon)$ (see also remark 5.4).

Actually, for some y the last summand in the expression (5.8) may be missing but this error can be bounded easily. Inserting (5.34) and (5.35) as well as (5.29)

into (5.27) we can conclude that in the case where $(t - s) > (\varepsilon z)^2$ and $z > 0$ we get

$$\left| \partial_x J_\varepsilon^z(s, t) - \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)}(y) G(u_\varepsilon(y)) du_\varepsilon(y) \right| \lesssim \frac{|\varepsilon z|^{3\alpha-1}}{(t-s)^{\frac{1+2\alpha}{2}} s^{\frac{\beta}{2}}}. \quad (5.36)$$

The case where

$$s > \varepsilon^2, \quad (t - s) > (\varepsilon z)^2, \quad \text{and} \quad z < 0 \quad (5.37)$$

can be treated in the same way by following the calculations from (5.25) to (5.36). The only difference is that in (5.27) the expression $\frac{1}{\varepsilon z} \int_0^{\varepsilon z}$ has to be replaced by $\frac{-1}{\varepsilon|z|} \int_0^{|\varepsilon z|}$. Then in order to derive (5.35), the second statement of Lemma 5.2 is used instead of the first. So in this case the bound (5.36) holds as well.

In the case where $s \leq \varepsilon^2$ or $(t - s) \leq (\varepsilon z)^2$ we brutally bound

$$\begin{aligned} \left| \partial_x J_\varepsilon^z(s, t) - \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon) du_\varepsilon \right| \\ \leq \left| \partial_x J_\varepsilon^z(s, t) \right| + \left| \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon) du_\varepsilon \right|. \end{aligned} \quad (5.38)$$

The first term can be bounded by

$$\begin{aligned} \left| \partial_x J_\varepsilon^z(s, t) \right| &\lesssim \frac{1}{\varepsilon|z|} \int_{-\pi}^{\pi} |\partial_x p_{t-s}(x - y)| dy \\ &\quad \left(|G|_0 |u_\varepsilon|_\alpha |\varepsilon z|^\alpha + |DG|_0 |u'_\varepsilon|_0^2 |\mathbf{X}_\varepsilon|_\alpha |\varepsilon z|^\alpha \right) \\ &\lesssim |\varepsilon z|^{\alpha-1} (t - s)^{-1/2}. \end{aligned} \quad (5.39)$$

The second integral can be bounded by applying the scaling Lemma A.5 to the functions f and Y as in (5.30). Using the bound (5.33) on the controlled rough path norms of Y one obtains

$$\left| \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon) du_\varepsilon \right| \lesssim (t - s)^{\frac{-2+\alpha}{2}} s^{-\frac{\beta}{2}}. \quad (5.40)$$

Actually, the bounds (5.39) and (5.40) also hold with α replaced by $\tilde{\alpha}$ and β replaced by $\tilde{\beta}$. We will prefer to work with this bound. This establishes the bounds for $\partial_x J_\varepsilon^z(s, t) - \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon(s)) du_\varepsilon$.

Finally, to get a bound on $\partial_x J^z(s, t; x)$ it remains to bound

$$\left| \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u(s)) du - \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon(s)) du_\varepsilon \right|. \quad (5.41)$$

Similar calculations have been performed in [HW10, Prop. 5.8] and we will only give an outline here. In order to get a better rate at this point we will work with the smaller Hölder exponent $\tilde{\alpha}$. Of course, all the bounds that we have imposed on the C^α norms of u, u_ε etc. remain valid for $\tilde{\alpha}$. In [HW10, Rem. 5.3] a version of the scaling lemma A.5 is given, that bounds the difference of two rescaled integrals. This yields an estimate for the expression in (5.41) in terms of $|X - X_\varepsilon|_{C^{\tilde{\alpha}}}$, $|\mathbf{X} - \mathbf{X}_\varepsilon|_{2\tilde{\alpha}}$, $|u - u_\varepsilon|_{C^{\tilde{\alpha}}}$, $|u' - u'_\varepsilon|_{C^{\tilde{\alpha}}}$, $|R_u - R_{u_\varepsilon}|_{\tilde{\alpha}}$, as well as the norms for the differences of $G(u) - G(u_\varepsilon)$. Bounds on these norms can be found in [HW10, Lem. 5.5].

Then one gets

$$\begin{aligned} & \left| \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u_\varepsilon(s)) du_\varepsilon - \int_{-\pi}^{\pi} \partial_x p_{c_f(t-s)} G(u(s)) du \right| \\ & \lesssim |G|_{\mathcal{C}^3}(t-s)^{\frac{-2+\tilde{\alpha}}{2}} s^{-\frac{\tilde{\beta}}{2}} \mathcal{D}_\varepsilon. \end{aligned} \quad (5.42)$$

Let us summarise (5.36), (5.38)- (5.40) and (5.42). If $(t-s) > (\varepsilon z)^2$ and $s > \varepsilon^2$ we get

$$\begin{aligned} \sup_x |\partial_x J^z(s, t; x)| & \lesssim |\varepsilon z|^{3\alpha-1} (t-s)^{\frac{-1-2\alpha}{2}} s^{-\frac{\beta}{2}} \\ & + (t-s)^{\frac{-2+\tilde{\alpha}}{2}} s^{-\frac{\tilde{\beta}}{2}} \mathcal{D}_\varepsilon. \end{aligned} \quad (5.43)$$

If $s < \varepsilon^2$ or $(t-s) \leq (\varepsilon z)^2$ we get

$$\begin{aligned} \sup_x |\partial_x J^z(s, t; x)| & \lesssim |\varepsilon z|^{\alpha-1} (t-s)^{-1/2} + (t-s)^{\frac{-2+\tilde{\alpha}}{2}} s^{-\frac{\tilde{\beta}}{2}} \\ & + (t-s)^{\frac{-2+\tilde{\alpha}}{2}} s^{-\frac{\tilde{\beta}}{2}} \mathcal{D}_\varepsilon. \end{aligned} \quad (5.44)$$

As stated above, we can repeat the entire argument with $\partial_x p_{c_f(t-s)}$ replaced by $p_{c_f(t-s)}$ and obtain very similar bounds for $\sup_x |J^z(s, t; x)|$. The only difference is that all the exponents of $(t-s)$ are increased by $1/2$ compared to those appearing in (5.43) and (5.44). Then we can interpolate between the \mathcal{C}^1 and the \mathcal{C} bounds to get bounds on J^z in an arbitrary \mathcal{C}^γ for $\gamma \in (0, 1)$. These norms are then bounded by exactly the same quantities as on the right-hand side of (5.43) and (5.44) with exponents of $(t-s)$ increased by $(1-\gamma)/2$.

To conclude it only remains to refer to the first statement in Lemma 6.4, that states that the operators $A_\varepsilon(t)$ are uniformly continuous from $\mathcal{C}^{\gamma+\kappa}$ to any \mathcal{C}^γ for any $\kappa > 0$. Then integrating (5.43) and (5.44) over s , and observing that for any $\beta_1, \beta_2 \in [0, 1)$ we have

$$\int_0^t (t-s)^{-\beta_1} s^{-\beta_2} ds = t^{1-\beta_1-\beta_2} \int_0^1 (1-s)^{-\beta_1} s^{-\beta_2} ds$$

we arrive at the desired conclusion. \square

We continue with a bound on I_2 .

Lemma 5.6. *Let $\tilde{\gamma} > \gamma \in (0, 1)$. Then, for any $\kappa > 0$ small enough and for any $t \in [0, T]$, we have*

$$|I_2(t)|_{\mathcal{C}^\gamma} \lesssim \varepsilon^{\tilde{\gamma}-\gamma-\kappa} t^{\frac{1-\tilde{\gamma}+\alpha-\beta}{2}}. \quad (5.45)$$

Proof. In the same way as in (5.40) we can see that

$$\left| \int_{-\pi}^{\pi} p_{c_f(t-s)}(\cdot - y) G(u(s, y)) dy \Psi(s, y) \right|_{\mathcal{C}^{\tilde{\gamma}}} \lesssim (t-s)^{\frac{-1-\tilde{\gamma}+\alpha}{2}} s^{-\frac{\beta}{2}}.$$

The desired bound (5.45) then follows immediately from (6.7). \square

To finish the proof of (5.6) it only remains to recall that according to (5.17) and (5.19) we have

$$|\Xi(t) - \Xi_\varepsilon(t)|_{\mathcal{C}^\gamma} \leq \int_{\mathbb{R}} (|I_1^z(t)|_{\mathcal{C}^\gamma} + |I_2(t)|_\gamma) |z| |\mu|(dz).$$

Combining (5.20) and (5.45) and recalling the existence of second moments for $|\mu|$ we get the desired bound. \square

We will now give the proof of the bound (5.7) concerning the time regularity.

Proof of (5.7). Fix times $\varepsilon^2 < s < t < T$. Our aim is to derive uniform in x bounds on the quantity

$$\left| (\Xi(t, x) - \Xi_\varepsilon(t, x)) - (\Xi(s, x) - \Xi_\varepsilon(s, x)) \right|. \quad (5.46)$$

We will use the same quantities J^z , J_0^z , and J_ε^z defined above in (5.21). Recall that (5.39), (5.43), and (5.44) we have already derived bounds on the \mathcal{C}^1 norm of these quantities. Furthermore, as in (5.40) we can see that

$$\left| J_0^z(s, t) \right|_{\mathcal{C}^\gamma} \lesssim (t - s)^{\frac{-1-\gamma+\alpha}{2}} s^{-\frac{\beta}{2}}. \quad (5.47)$$

With these bounds at hand we can start to give the estimate on (5.46). As before we will omit the space variable x in the notation. Similarly to above in (5.17) and (5.19) we write

$$(\Xi(t, x) - \Xi_\varepsilon(t, x)) - (\Xi(s, x) - \Xi_\varepsilon(s, x)) = \int_{\mathbb{R}} z \sum_{j=1}^4 I_j^z(s, t) \mu(dz),$$

where the I_j^z are given as the following integrals over $[\varepsilon^2, s]$:

$$\begin{aligned} I_1(s, t) &= \int_{\varepsilon^2}^s \left[(S(t-s) - Id) A(s-\tau) \right. \\ &\quad \left. - (S_\varepsilon(t-s) - Id) A_\varepsilon(s-\tau) \right] J_0(\tau, s) d\tau, \\ I_2^z(s, t) &= \int_{\varepsilon^2}^s (S_\varepsilon(t-s) - Id) A_\varepsilon(s-\tau) J^z(\tau, s) d\tau, \end{aligned} \quad (5.48)$$

and the following integrals over $[s, t]$:

$$\begin{aligned} I_3(s, t) &= \int_s^t [A(t-\tau) - A_\varepsilon(t-\tau)] J_0(\tau, t) d\tau, \\ I_4^z(s, t) &= \int_s^t A_\varepsilon(t-\tau) [J^z(\tau, t)] d\tau. \end{aligned}$$

We shall show that the bounds on these quantities follow in a straightforward way from Lemma 6.4 and the bounds stated above.

Combining (5.47) and Lemma 6.6 we get for any $\gamma \in (0, 1)$ and $\kappa > 0$ small enough

$$\begin{aligned} |I_1(s, t)| &\lesssim \varepsilon^{1-\gamma-\kappa} (t-s)^{\frac{\gamma}{2}} \int_{\varepsilon^2}^s (s-\tau)^{\frac{-2+\alpha}{2}} \tau^{-\frac{\beta}{2}} d\tau \\ &\lesssim \varepsilon^{1-\gamma-\kappa} (t-s)^{\frac{\gamma}{2}}. \end{aligned} \quad (5.49)$$

Here we have used the fact that $\beta < \alpha$. The constant in the last line depends on the final time T . For I_2^z , combining Lemma 6.4 as well as (5.43) and (5.44), we get for any $\gamma < 2 - 2\alpha - \beta$,

$$\begin{aligned} |I_2^z(s, t)| &\lesssim (t-s)^{\frac{\gamma}{2}} \left[\mathcal{D}_\varepsilon s^{\frac{1+\tilde{\alpha}-\gamma-\tilde{\beta}-\kappa}{2}} \right. \\ &\quad \left. + |\varepsilon z|^{3\alpha-1} s^{1-\frac{2\alpha+\gamma+\beta+\kappa}{2}} + |\varepsilon z|^{1+\tilde{\alpha}-\tilde{\beta}-\gamma-\kappa} \right]. \end{aligned}$$

To bound I_3 we use Lemma 6.4 and (5.47) and get that

$$|I_3(s, t)| \lesssim \varepsilon^{1+\alpha-\beta-\gamma-\kappa} (t-s)^{\frac{\gamma}{2}}. \quad (5.50)$$

Finally, for I_4^z we get using Lemma 6.4 and (5.43), (5.44) that for any $\gamma < 2 - 2\alpha - \beta$

$$\begin{aligned} |I_4^z(s, t)| &\lesssim (t-s)^{\frac{\gamma}{2}} \left[\mathcal{D}_\varepsilon t^{\frac{1+\bar{\alpha}-\bar{\beta}-\gamma-\kappa}{2}} + |\varepsilon z|^{3\alpha-1} t^{1-\frac{2\alpha-\beta-\gamma-\kappa}{2}} \right] \\ &\quad + \int_s^t \mathbf{1}_{(t-s) \leq (\varepsilon z)^2} \left[|\varepsilon z|^{\bar{\alpha}-1} (t-\tau)^{-\kappa} + (t-s)^{\frac{-\kappa-2\bar{\alpha}}{2}} \tau^{-\frac{\bar{\beta}}{2}} \right] d\tau. \end{aligned} \quad (5.51)$$

The integral on the right-hand side of (5.51) can be bounded by

$$|\varepsilon z|^{1+\alpha-2\kappa} \wedge |\varepsilon z|^{\bar{\alpha}-1} (t-s)^{1-\kappa} + |\varepsilon z|^{2-2\bar{\alpha}-\bar{\beta}-\kappa} \wedge (t-s)^{\frac{2-2\bar{\alpha}-\bar{\beta}-\kappa}{2}}.$$

By interpolation (and redefining κ) one can bound this by

$$(t-s)^{\frac{\gamma}{2}} \left[|\varepsilon z|^{1+\bar{\alpha}-\gamma-\kappa} + |\varepsilon z|^{2-2\bar{\alpha}-\bar{\beta}-\gamma-\kappa} \right].$$

Then summarising these bounds and integrating over z we obtain the desired bound (5.7). \square

6. BOUNDS ON THE APPROXIMATED SEMIGROUP

Throughout the paper, we frequently need bounds on the approximated heat semigroup S_ε . The following Lemmas 6.1 and 6.2 will be necessary to proof Lemma 3.5 on the regularity of the remainder term R_ε^θ . The other main results of this section, Lemma 6.4 and 6.6, state that the regularising properties on Hölder spaces of the approximated heat semigroup $S_\varepsilon(t)$ are similar to those of $S(t)$.

The following result is a special case of [Ste57] adapted to the periodic setting. For the convenience of the reader we include a self-contained proof. In this lemma we write L_α^2 to denote the weighted L^2 -space endowed with the norm

$$|f|_{L_\alpha^2} := \left(\int_{-\pi}^{\pi} |f(x)|^2 |x|^{2\alpha} dx \right)^{\frac{1}{2}}.$$

Lemma 6.1. *Let $a : [-\pi, \pi] \rightarrow \mathbb{R}$ be an L^∞ function, and consider the integral operator $A : L^2 \rightarrow L^2$ defined by*

$$Af(x) = \int_{-\pi}^{\pi} a(x-y) f(y) dy,$$

where a is extended periodically. Suppose that there exist constants $C_1, C_2 > 0$ such that for all $f \in L^2$ and $x \in [-\pi, \pi]$

$$|Af|_{L^2} \leq C_1 |f|_{L^2}, \quad |a(x)| \leq \frac{C_2}{|x|}. \quad (6.1)$$

Then, for $\alpha \in (0, \frac{1}{2})$ there exists a constant $C > 0$ depending only on C_1, C_2 , and α , such that the bound

$$|Af|_{L_\alpha^2} \leq C |f|_{L_\alpha^2}$$

holds for all $f \in L_\alpha^2$.

Proof. The proof of this results consists of two parts. First we give a proof of the analogous result, in which the torus $[-\pi, \pi]$ is replaced by \mathbb{R} ; this result is due to Stein [Ste57] and we provide his proof for the convenience of the reader. In the second part we use the result on \mathbb{R} to obtain the desired result on the torus.

Step 1. Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be an L^∞ function, and consider the integral operator $A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by

$$Af(x) = \int_{\mathbb{R}} a(x-y)f(y) dy .$$

Suppose that there exist constants $C_1, C_2 > 0$ such that for all $f \in L^2$ and $x \in \mathbb{R}$,

$$|Af|_{L^2(\mathbb{R})} \leq C_1 |f|_{L^2(\mathbb{R})} , \quad |a(x)| \leq \frac{C_2}{|x|} . \quad (6.2)$$

Then, we claim that for $\alpha \in (0, \frac{1}{2})$ there exists a constant $C_\alpha > 0$ depending only on α , such that the bound

$$|Af|_{L_\alpha^2(\mathbb{R})} \leq (C_1 + C_2 C_\alpha) |f|_{L_\alpha^2(\mathbb{R})}$$

holds for all $f \in L_\alpha^2(\mathbb{R})$. Here $L_\alpha^2(\mathbb{R})$ denotes the weighted L^2 -space endowed with the norm

$$|f|_{L_\alpha^2(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha} dx \right)^{\frac{1}{2}} .$$

In order to prove the claim, let M_α denote the multiplication operator defined by $M_\alpha f(x) := |x|^\alpha f(x)$ and consider the commutator $[M_\alpha, A] := M_\alpha A - A M_\alpha$. Then we have

$$|Af|_{L_\alpha^2(\mathbb{R})} = |M_\alpha Af|_{L^2(\mathbb{R})} \leq |[M_\alpha, A]f|_{L^2(\mathbb{R})} + |A M_\alpha f|_{L^2(\mathbb{R})} .$$

Since (6.2) implies that

$$|A M_\alpha f|_{L^2(\mathbb{R})} \leq C_1 |M_\alpha f|_{L^2(\mathbb{R})} = C_1 |f|_{L_\alpha^2(\mathbb{R})} ,$$

it remains to estimate the commutator $[M_\alpha, A]$. For this purpose we set $k(x, y) := \frac{|1-|x/y|^\alpha|}{|x-y|}$ and consider the operator K defined by

$$Kf(x) = \int_{\mathbb{R}} k(x, y)f(y) dy .$$

It follows from (6.2) that

$$\begin{aligned} |[M_\alpha, A]f(x)| &= \left| \int_{\mathbb{R}} a(x-y)(1-|x/y|^\alpha)|y|^\alpha f(y) dy \right| \\ &\leq C_2 \int_{\mathbb{R}} k(x, y)(M_\alpha |f|)(y) dy = C_2 K(M_\alpha |f|)(x) . \end{aligned} \quad (6.3)$$

Since for $\lambda \in \mathbb{R} \setminus \{1\}$,

$$|xk(x, \lambda x)| = \left| \frac{1-|\lambda|^{-\alpha}}{1-\lambda} \right| ,$$

a change of variables yields

$$|Kf(x)| \leq \int_{\mathbb{R}} |xk(x, \lambda x)f(\lambda x)| d\lambda = \int_{\mathbb{R}} \left| \frac{1-|\lambda|^{-\alpha}}{1-\lambda} f(\lambda x) \right| d\lambda .$$

Hence by Minkowski's integral inequality we get

$$\begin{aligned} |Kf|_{L^2(\mathbb{R})} &\leq \int_{\mathbb{R}} \left| \frac{1-|\lambda|^{-\alpha}}{1-\lambda} \right| \left(\int_{\mathbb{R}} |f(\lambda x)|^2 dx \right)^{\frac{1}{2}} d\lambda \\ &\leq |f|_{L^2(\mathbb{R})} \int_{\mathbb{R}} \left| \frac{1-|\lambda|^{-\alpha}}{|\lambda|^{\frac{1}{2}}(1-\lambda)} \right| d\lambda =: C_{\alpha} |f|_{L^2(\mathbb{R})} \end{aligned}$$

with $C_{\alpha} < \infty$. By (6.3) we thus obtain

$$\begin{aligned} |[M_{\alpha}, A]f|_{L^2(\mathbb{R})} &\leq C_2 |K(M_{\alpha}|f|)|_{L^2(\mathbb{R})} \\ &\leq C_2 C_{\alpha} |M_{\alpha}|f||_{L^2(\mathbb{R})} = C_2 C_{\alpha} |f|_{L^2_{\alpha}(\mathbb{R})}, \end{aligned}$$

which completes the proof of the claim.

Step 2. We shall now use Step 1 to prove the corresponding assertion on the torus. Thus, let a be as in the statement of the result, fix a small constant $\delta \in (0, \frac{\pi}{2})$, and consider the functions $b, g : [-\pi, \pi] \rightarrow \mathbb{R}$ defined by

$$b \stackrel{\text{def}}{=} a \mathbf{1}_{(-\delta, \delta)}, \quad g \stackrel{\text{def}}{=} a \mathbf{1}_{(-\delta, \delta)^c},$$

so that b contains the “bad” part of a and g contains the “good” (i.e. bounded) part of a . Furthermore, for any bounded function $a : [-\pi, \pi] \rightarrow \mathbb{R}$, we define operators T_a and \tilde{T}_a , acting on functions defined on $[-\pi, \pi]$ and \mathbb{R} respectively, by

$$T_a f(x) = \int_{-\pi}^{\pi} a(y) P f(x-y) dy, \quad \tilde{T}_a f(x) = \int_{-\pi}^{\pi} a(y) f(x-y) dy,$$

where Pf denotes the periodic extension of f . For $f \in L^2_{\alpha}[-\pi, \pi]$ we define $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\tilde{f} = (Pf) \mathbf{1}_{[-\pi-\delta, \pi+\delta]}.$$

Note that $|\tilde{f}|^2_{L^2_{\alpha}(\mathbb{R})} \lesssim |f|^2_{L^2_{\alpha}[-\pi, \pi]}$ and that $\tilde{T}_b \tilde{f}(x) = T_b f(x)$ for $x \in [-\pi, \pi]$, so that

$$|T_b f|_{L^2_{\alpha}[-\pi, \pi]} \leq |\tilde{T}_b \tilde{f}|_{L^2_{\alpha}(\mathbb{R})}.$$

To show that T_b is bounded on $L^2_{\alpha}[-\pi, \pi]$ it thus suffices to show that \tilde{T}_b is bounded on $L^2_{\alpha}(\mathbb{R})$. To show this, we need to check the conditions of Step 1. Clearly, $|b(x)| \lesssim 1/|x|$ for all $x \in \mathbb{R}$ by the assumption on a . It thus remains to show that \tilde{T}_b is bounded as an operator on $L^2(\mathbb{R})$.

For this purpose, take $\varphi \in L^2(\mathbb{R})$ and for $j \in \mathbb{Z}$ define $\varphi_j : [-\pi, \pi] \rightarrow \mathbb{R}$ by $\varphi_j(x) = \varphi(x + j\pi)$. For $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we then have the identity $(\tilde{T}_b \varphi)(x + j\pi) = T_b \varphi_j(x)$. Since T_a is bounded on $L^2[-\pi, \pi]$ by assumption and T_g is bounded on the same space since $g \in L^1[-\pi, \pi]$, it follows that T_b is bounded on $L^2[-\pi, \pi]$ as well, and its operator norm N_{δ} depends only on δ . We infer that

$$\begin{aligned} |\tilde{T}_b \varphi|_{L^2(\mathbb{R})}^2 &= \sum_{j \in \mathbb{Z}} |(\tilde{T}_b \varphi) \mathbf{1}_{[(j-\frac{1}{2})\pi, (j+\frac{1}{2})\pi]}|_{L^2(\mathbb{R})}^2 \\ &= \sum_{j \in \mathbb{Z}} |(T_b \varphi_j) \mathbf{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}|_{L^2[-\pi, \pi]}^2 \leq \sum_{j \in \mathbb{Z}} |T_b \varphi_j|_{L^2[-\pi, \pi]}^2 \\ &\leq N_{\delta}^2 \sum_{j \in \mathbb{Z}} |\varphi_j|_{L^2[-\pi, \pi]}^2 = 2N_{\delta}^2 |\varphi|_{L^2(\mathbb{R})}^2. \end{aligned}$$

The claim now follows from Step 1.

It remains to show that T_g is bounded on L_α^2 . This however follows from the fact that

$$|x|^\alpha T_g f(x) = \int_{-\pi}^{\pi} \frac{|x|^\alpha}{|y|^\alpha} (Pg)(x-y) |y|^\alpha f(y) dy .$$

Indeed, it suffices to note that g (and therefore its periodic continuation Pg) is bounded and that, since $\alpha < \frac{1}{2}$, the function $|x|^\alpha/|y|^\alpha$ is square-integrable. The Cauchy-Schwartz inequality immediately implies that the L^2 norm of the left hand side is bounded by $\|f\|_{L_\alpha^2}$ as required. \square

Lemma 6.2. *Let $a: [-\pi, \pi] \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function with Fourier coefficients $\widehat{a}(k)$, $k \in \mathbb{Z}$. Assume that*

$$\sum_{k \in \mathbb{Z}} |\widehat{a}(k) - \widehat{a}(k-1)| \leq K .$$

Then, for any $x \in [-\pi, \pi]$ one has

$$|a(x)| \leq \frac{\pi K}{2|x|} .$$

Proof. Using the identity $|1 - e^{ix}| = 2|\sin(x/2)|$ we obtain for all $x \in [-\pi, \pi]$,

$$\begin{aligned} 2\pi^{-1} |xa(x)| &\leq 2|a(x) \sin(x/2)| = \left| \sum_{k \in \mathbb{Z}} \widehat{a}(k) e^{ikx} (1 - e^{ix}) \right| \\ &= \left| \sum_{k \in \mathbb{Z}} \widehat{a}(k) e^{ikx} - \sum_{k \in \mathbb{Z}} \widehat{a}(k) e^{i(k+1)x} \right| \\ &= \left| \sum_{k \in \mathbb{Z}} (\widehat{a}(k) - \widehat{a}(k-1)) e^{ikx} \right| \leq \sum_{k \in \mathbb{Z}} |\widehat{a}(k) - \widehat{a}(k-1)| \leq K , \end{aligned}$$

which is the required bound. \square

We state the regularising properties of S_ε in terms of the operator A_ε . Recall that

$$S_\varepsilon(t) = A_\varepsilon(t) S(c_f t) ,$$

and that A_ε is the Fourier multiplier associated to the sequence of functions

$$a_{\varepsilon,t}(k) = \exp \left(-k^2(1 - c_f)t \bar{f}(\varepsilon k) \right) .$$

Here, and in the sequel we will often write $\bar{f} = \frac{f - c_f}{1 - c_f}$ to shorten the notation. Recall that \bar{f} is bounded from below by $c_f > 0$ according to Assumption 1.1, \bar{f} is continuously differentiable on $[-\delta, \delta]$ with bounded derivative, and $\bar{f}(0) = 1$. We also recall the notation

$$a_t(k) = \exp \left(-k^2(1 - c_f)t \right) .$$

In order to obtain the desired bounds on $A_\varepsilon(t)$ as an operator on Hölder spaces, we first state a simple corollary of the Marcinkiewicz multiplier theorem [Mar39]. Note that we cannot apply standard multiplier results in Hölder spaces such as the one in [ABB04], since in our application the conditions in these spaces are not satisfied uniformly in ε . In order to state the result we introduce the following notation. For any sequence $\{m(k)\}_{k \in \mathbb{Z}}$ we define

$$\|m\|_{\mathcal{M}} := \sup_{k \in \mathbb{Z}} |m(k)| + \sup_{l \geq 0} \sum_{k=2^l}^{2^{l+1}-1} \sum_{\sigma \in \{-1,1\}} |m(\sigma k) - m(\sigma(k+1))| . \quad (6.4)$$

The result can now be formulated as follows.

Lemma 6.3. *Let $\{m(k)\}_{k \in \mathbb{Z}}$ be a real sequence and let T_m be the associated Fourier multiplication operator given by*

$$T_m e^{ik \cdot} = m(k) e^{ik \cdot}.$$

For any $\gamma > 0$ we define the sequence $\{m^\gamma\}_{k \in \mathbb{Z}}$ by

$$m^\gamma(k) = |k|^{-\gamma} m(k) \quad (6.5)$$

for $k \neq 0$ and $m^\gamma(0) = m(0)$. Then, for any $\bar{\gamma} > 0$ and any $\kappa > 0$ we have

$$\|T_m\|_{\mathcal{C}^{\bar{\gamma}+\gamma} \rightarrow \mathcal{C}^{\bar{\gamma}-\kappa}} \lesssim \|m^\gamma\|_{\mathcal{M}}.$$

Proof. For any $1 < p < \infty$ the Marcinkiewicz multiplier theorem [Mar39] asserts that

$$\|T_{m^\gamma}\|_{L^p \rightarrow L^p} \lesssim \|m^\gamma\|_{\mathcal{M}},$$

where T_{m^γ} is the Fourier multiplier associated to m^γ . Hence, it follows immediately from the definition of the Bessel potential spaces $H^{\gamma,p}$ that

$$\|T_m\|_{H^{\gamma+\bar{\gamma}-\kappa/2,p} \rightarrow H^{\bar{\gamma}-\kappa/2,p}} \lesssim \|m^\gamma\|_{\mathcal{M}}.$$

(See, e.g. [Gra09, Section 6.1.2] or [Mey92] for proofs in the whole space; the extension to the torus is immediate.) Then the desired statement follows from the embedding

$$\mathcal{C}^{\gamma+\bar{\gamma}} \hookrightarrow H^{\gamma+\bar{\gamma}-\kappa/2,p},$$

and the Sobolev embedding

$$H^{\bar{\gamma}-\kappa/2,p} \hookrightarrow \mathcal{C}^{\bar{\gamma}-\kappa},$$

which holds as soon as p is sufficiently large. \square

Now we are ready to apply this result to the operators $A_\varepsilon(t)$.

Lemma 6.4. *Let $0 < \bar{\gamma} < 1$ and $\kappa > 0$. Then the bound*

$$\sup_{t \in [0,T]} \sup_{\varepsilon \in (0,1)} \|A_\varepsilon(t)\|_{\mathcal{C}^{\bar{\gamma}} \rightarrow \mathcal{C}^{\bar{\gamma}-\kappa}} \lesssim 1. \quad (6.6)$$

holds. Furthermore, for any $\gamma \in [0, 1]$, we have

$$\sup_{t \in [0,T]} \|A(t) - A_\varepsilon(t)\|_{\mathcal{C}^{\bar{\gamma}+\gamma} \rightarrow \mathcal{C}^{\bar{\gamma}-\kappa}} \lesssim \varepsilon^\gamma. \quad (6.7)$$

Proof. The first bound follows immediately from Lemma 6.3 observing that

$$\|a_{\varepsilon,t}\|_{\mathcal{M}} \leq 1 + 2\|a_{\varepsilon,t}\|_{BV},$$

which is uniformly bounded by Assumption 1.2.

As a shorthand, we use the notations

$$\delta_\varepsilon a_t \stackrel{\text{def}}{=} a_{\varepsilon,t} - a_t, \quad \delta_\varepsilon a_t^\gamma \stackrel{\text{def}}{=} a_{\varepsilon,t}^\gamma - a_t^\gamma,$$

where $a_{\varepsilon,t}^\gamma$ and a_t^γ are defined as in (6.5). The second bound follows from 6.3 as soon as we have established the estimate

$$\|\delta_\varepsilon a_t^\gamma\|_{\mathcal{M}} \lesssim \varepsilon^\gamma. \quad (6.8)$$

Observe that $\delta_\varepsilon a_t^\gamma(0) = 0$, so that from now on we will only deal with $k \neq 0$. Also recall that according to Assumption 1.1 f is differentiable on $(-\delta, \delta)$ with bounded derivatives.

In order to establish (6.8) we start by showing that

$$\sup_{k \in \mathbb{Z}} |\delta_\varepsilon a_t^\gamma(k)| \lesssim \varepsilon^\gamma. \quad (6.9)$$

In order to keep the formulas short, we will always drop the prefactor $(1 - c_f)$ in the definitions of $a_{\varepsilon,t}$ and a_t . This corresponds to a rescaling of the time variable, which does not change the statement we want to prove. Furthermore, by symmetry it suffices to consider the positive Fourier modes, hence, to simplify notation, we will neglect the terms with $\sigma = -1$ in the definition of $\|\cdot\|_{\mathcal{M}}$.

If $0 < |\varepsilon k| \leq \delta$ we can write

$$\begin{aligned} |\delta_\varepsilon a_t^\gamma(k)| &= \frac{1}{|k|^\gamma} \left[\exp(-tk^2 \bar{f}(\varepsilon k)) - \exp(-tk^2) \right] \\ &\lesssim \frac{1}{|k|^\gamma} \exp(-c_f tk^2) tk^2 |\bar{f}(\varepsilon k) - 1| \lesssim \frac{1}{|k|^\gamma} |\varepsilon k| \lesssim \varepsilon^\gamma. \end{aligned} \quad (6.10)$$

Here we have made use of the fact that the function $x \mapsto x \exp(-c_f x)$ is bounded on $[0, \infty)$ as well as of the boundedness of \bar{f}' on $(-\delta, \delta)$.

If $|\varepsilon k| \geq \delta$ the bound (6.9) can be established simply by writing

$$|\delta_\varepsilon a_t^\gamma(k)| \lesssim |k|^{-\gamma} \lesssim \varepsilon^\gamma. \quad (6.11)$$

The bounds on the BV-norms of the Paley-Littlewood blocks

$$\sum_{k=2^l}^{2^{l+1}-1} |\delta_\varepsilon a_t^\gamma(k) - \delta_\varepsilon a_t^\gamma(k+1)|$$

require more thought. Actually, we can always write using the inequality $|fg|_{\text{BV}} \leq |f|_{\text{BV}} |g|_{\text{BV}} + |g|_{\text{BV}} |f|_{\text{BV}}$

$$\begin{aligned} \sum_{k=2^l}^{2^{l+1}-1} |\delta_\varepsilon a_t^\gamma(k) - \delta_\varepsilon a_t^\gamma(k+1)| &\leq \frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |\delta_\varepsilon a_t(k) - \delta_\varepsilon a_t(k+1)| \\ &\quad + \frac{1}{2^{l\gamma}} \sup_{k \in [2^l, 2^{l+1}]} |\delta_\varepsilon a_t(k)|. \end{aligned} \quad (6.12)$$

The second summand can be bounded as in (6.10) and (6.11). We get

$$\frac{1}{2^{l\gamma}} \sup_{k \in [2^l, 2^{l+1}]} |\delta_\varepsilon a_t(k)| \lesssim \varepsilon^\gamma. \quad (6.13)$$

For the first term on the right-hand side of (6.12) we distinguish between different cases.

We first consider the case where $\varepsilon 2^{l+1} \geq \delta$. In this case the $a_{\varepsilon,t}(k)$ for $k \in [2^l, 2^{l+1}]$ are not good approximations to the $a_t(k)$. Hence we bound the difference by the sum

$$|\delta_\varepsilon a_t(k) - \delta_\varepsilon a_t(k+1)| \leq |a_{\varepsilon,t}(k) - a_{\varepsilon,t}(k+1)| + |a_t(k) - a_t(k+1)|. \quad (6.14)$$

Then we get using Assumption 1.2 on the boundedness of the BV norm of $a_{\varepsilon,t}$

$$\frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |a_{\varepsilon,t}(k) - a_{\varepsilon,t}(k+1)| \lesssim \frac{1}{2^{l\gamma}} \lesssim \varepsilon^\gamma. \quad (6.15)$$

The second term on the right-hand side of (6.14) can be bounded in the same way.

Secondly, we consider the case $\varepsilon 2^{l+1} < \delta$. In order to treat this case, we claim that for any non-negative numbers g_{ij} with $i, j \in \{0, 1\}$, we have

$$\begin{aligned} & |e^{-g_{00}} - e^{-g_{01}} - e^{-g_{10}} + e^{-g_{11}}| \\ & \leq e^{-a} \left[|g_{00} - g_{01} - g_{10} + g_{11}| \right. \\ & \quad \left. + \left(|g_{00} - g_{01}| + |g_{10} - g_{11}| \right) \left(|g_{00} - g_{10}| + |g_{01} - g_{11}| \right) \right], \end{aligned} \quad (6.16)$$

where $m = \min g_{ij}$. To see this, set

$$g(\lambda, \mu) = (1 - \lambda)(1 - \mu)g_{00} + (1 - \lambda)\mu g_{01} + \lambda(1 - \mu)g_{10} + \lambda\mu g_{11}$$

and note that the left-hand side of (6.16) can be written as

$$\begin{aligned} & \left| \int_0^1 \int_0^1 \partial_\lambda \partial_\mu \exp(-g(\lambda, \mu)) d\lambda d\mu \right| \\ & \leq \left| \int_0^1 \int_0^1 \left[|\partial_\lambda \partial_\mu g(\lambda, \mu)| + |\partial_\lambda g(\lambda, \mu)| |\partial_\mu g(\lambda, \mu)| \right] \exp(-g(\lambda, \mu)) d\lambda d\mu \right|. \end{aligned}$$

The estimate (6.16) follows using the inequalities

$$\begin{aligned} \partial_\lambda \partial_\mu g(\lambda, \mu) &= g_{00} - g_{01} - g_{10} + g_{11}, \\ \partial_\lambda g(\lambda, \mu) &\leq |g_{00} - g_{10}| + |g_{01} - g_{11}|, \\ \partial_\mu g(\lambda, \mu) &\leq |g_{00} - g_{01}| + |g_{10} - g_{11}|, \\ g(\lambda, \mu) &\geq m. \end{aligned}$$

Applying this estimate to $g_{0i} = (k + i)^2 t$ and $g_{1i} = (k + i)^2 t \bar{f}(\varepsilon(k + i))$, we infer that

$$\frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |\delta_\varepsilon a_t(k) - \delta_\varepsilon a_t(k+1)| \leq \frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} e^{-c_f t k^2} (B_{\varepsilon,t}(k) + C_{\varepsilon,t}(k)) \quad (6.17)$$

where

$$B_{\varepsilon,t}(k) = \left| t k^2 (\bar{f}(\varepsilon k) - 1) - t(k+1)^2 (\bar{f}(\varepsilon(k+1)) - 1) \right| \lesssim t \varepsilon k^2,$$

and, taking into account that $\varepsilon k \lesssim 1$,

$$\begin{aligned} C_{\varepsilon,t}(k) &= \left[\left| t k^2 - t(k+1)^2 \right| + \left| t k^2 f(\varepsilon k) - t(k+1)^2 f(\varepsilon(k+1)) \right| \right] \\ &\quad \times \left[\left| t k^2 (\bar{f}(\varepsilon k) - 1) \right| + t(k+1)^2 |\bar{f}(\varepsilon(k+1)) - 1| \right] \\ &\lesssim (t k) \cdot (t \varepsilon k^3) \\ &= t^2 \varepsilon k^4. \end{aligned}$$

Using these bounds, together with the fact that $M = \sup_{x \geq 0} \{x e^{-x}, x^2 e^{-x}\} < \infty$, we infer that

$$\frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |\delta_\varepsilon a_t(k) - \delta_\varepsilon a_t(k+1)| \lesssim \frac{M}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} \varepsilon \lesssim 2^{l(1-\gamma)} \varepsilon \lesssim \varepsilon^\gamma.$$

This finishes the proof of (6.8) and hence of (6.7). \square

The following result is now an immediate consequence.

Lemma 6.5. *Let $\lambda \in [0, 1]$ and $\alpha < \gamma + \lambda$. For $\kappa > 0$ sufficiently small,*

$$|S(t) - S_\varepsilon(t)|_{\mathcal{C}^\alpha \rightarrow \mathcal{C}^\gamma} \lesssim t^{-\frac{1}{2}(\gamma - \alpha + \lambda)} \varepsilon^{\lambda - \kappa}, \quad (6.18)$$

Proof. This follows from the decomposition

$$S(t) - S_\varepsilon(t) = S(c_f t)(A(t) - A_\varepsilon(t)),$$

using Lemma 6.4 and the standard regularisation properties of the heat semigroup. \square

The next result concerns the time regularity of solutions to the approximated heat equation. Recall that the approximated heat semigroup S_ε is not strongly continuous at 0 and we cannot expect convergence to zero of, say, $\|S_\varepsilon(t) - \text{Id}\|_{\mathcal{C}^\gamma \rightarrow \mathcal{C}}$ as $t \rightarrow 0$. However, the following result states that the approximating semigroup has nice time continuity properties for times $\geq \varepsilon^2$.

Lemma 6.6. *Let $\gamma \in [0, 2]$. Then, for all $t, s \geq \varepsilon^2$ and $u \in C^{\gamma + \kappa}$ we have*

$$\|(S_\varepsilon(t) - S_\varepsilon(s))u\|_{\mathcal{C}} \lesssim |t - s|^{\gamma/2} \|u\|_{\mathcal{C}^{\gamma + \kappa}}.$$

Proof. Without loss of generality we shall assume that $s \leq t$. Let us write

$$m_{\varepsilon, t}^\gamma(k) = k^{-\gamma} \exp(-tk^2 f(\varepsilon k)), \quad m_{\varepsilon, s}^\gamma(k) = \exp(-tk^2 f(\varepsilon k)).$$

Lemma 6.3 implies the desired result as soon as we have established that

$$\|m_{\varepsilon, t}^\gamma - m_{\varepsilon, s}^\gamma\|_{\mathcal{M}} \lesssim (t - s)^{\gamma/2},$$

where the norm $\|\cdot\|_{\mathcal{M}}$ has been defined in (6.4). By symmetry it suffices to consider the Fourier coefficients with $k > 0$, i.e. the terms with $\sigma = 1$. In order to shorten the notations, we will write

$$\delta m_{s, t}^{\gamma, \varepsilon}(k) \stackrel{\text{def}}{=} m_{\varepsilon, t}^\gamma(k) - m_{\varepsilon, s}^\gamma(k),$$

and similarly for $\delta m_{s, t}^\varepsilon$.

Using the boundedness of the function $x \mapsto f(x)^{\gamma/2} \exp(-x^2 f(x))$ and the fact that $\varepsilon^2 \leq s$, we obtain

$$\begin{aligned} \sup_{k \in \mathbb{Z}} |\delta m_{s, t}^{\gamma, \varepsilon}(k)| &\lesssim k^{-\gamma} \exp(-sk^2 f(\varepsilon k)) \left| 1 - \exp(-(t - s)k^2 f(\varepsilon k)) \right| \\ &\lesssim \exp(-\varepsilon^2 k^2 f(\varepsilon k)) (t - s)^{\gamma/2} f(\varepsilon k)^{\gamma/2} \\ &\lesssim (t - s)^{\gamma/2}. \end{aligned}$$

It remains to bound the Paley-Littlewood blocks, for which we will show that

$$\sum_{k=2^l}^{2^{l+1}-1} |\delta m_{s, t}^{\gamma, \varepsilon}(k) - \delta m_{s, t}^{\gamma, \varepsilon}(k + 1)| \lesssim |t - s|^{\gamma/2}. \quad (6.19)$$

We start with the case $0 < \varepsilon 2^{l+1} \leq \delta$, where the small constant δ has been defined in Assumption 1.1. On $[0, \delta]$, the function f is \mathcal{C}^1 by assumption, so that

$$\begin{aligned} \sum_{k=2^l}^{2^{l+1}-1} |\delta m_{s, t}^{\gamma, \varepsilon}(k) - \delta m_{s, t}^{\gamma, \varepsilon}(k + 1)| &\leq \int_{2^l}^{2^{l+1}} |\partial_x \delta m_{s, t}^{\gamma, \varepsilon}(x)| dx \\ &\leq \int_{2^l}^{2^{l+1}} \left| [\varepsilon(t - s)x^{2-\gamma} f'(\varepsilon x) + 2x^{1-\gamma}(t - s)f(\varepsilon x)] e^{-tx^2 f(\varepsilon x)} \right| dx \end{aligned}$$

$$\begin{aligned}
& + \int_{2^l}^{2^{l+1}} \left| \frac{\varepsilon s x^3 f'(\varepsilon x) + 2s x^2 f(\varepsilon x) + \gamma}{x^{1+\gamma}} \right| \left(e^{-s x^2 f(\varepsilon x)} - e^{-t x^2 f(\varepsilon x)} \right) dx \\
& \lesssim |t-s| \int_{2^l}^{2^{l+1}} x^{1-\gamma} e^{-t x^2 f(\varepsilon x)} dx \\
& + \int_{2^l}^{2^{l+1}} \frac{s x^2 + 1}{x^{1+\gamma}} \left(e^{-s x^2 f(\varepsilon x)} - e^{-t x^2 f(\varepsilon x)} \right) dx.
\end{aligned} \tag{6.20}$$

For the first term, we use the boundedness of f on $[0, \delta]$ as well as the lower bound $f \geq 2c_f$, so that

$$\begin{aligned}
|t-s| \int_{2^l}^{2^{l+1}} \left| x^{1-\gamma} e^{-t x^2 f(\varepsilon x)} \right| dx & \leq |t-s|^{\gamma/2} \int_{|t-s|^{1/2} 2^l}^{|t-s|^{1/2} 2^{l+1}} z^{1-\gamma} e^{-2z^2 c_f} dz \\
& \lesssim |t-s|^{\gamma/2},
\end{aligned}$$

as required, where we used the fact that $|t-s| \leq t$. We break the second term in two components. For the first one, we have

$$\begin{aligned}
& \int_{2^l}^{2^{l+1}} \frac{s x^2}{x^{1+\gamma}} \left(e^{-s x^2 f(\varepsilon x)} - e^{-t x^2 f(\varepsilon x)} \right) dx \\
& \lesssim \int_{2^l}^{2^{l+1}} s x^{1-\gamma} e^{-2s x^2 c_f} |1 - e^{-(t-s) x^2 f(\varepsilon x)}| dx \\
& \lesssim \int_{2^l}^{2^{l+1}} s x^{1-\gamma} e^{-2s x^2 c_f} |(t-s) x^2 f(\varepsilon x)|^{\gamma/2} dx \\
& \lesssim (t-s)^{\gamma/2} \int_0^\infty z e^{-2z^2 c_f} dz \lesssim (t-s)^{\gamma/2}.
\end{aligned}$$

For the remaining term, we obtain

$$\begin{aligned}
& \int_{2^l}^{2^{l+1}} \frac{1}{x^{1+\gamma}} \left(e^{-s x^2 f(\varepsilon x)} - e^{-t x^2 f(\varepsilon x)} \right) dx \\
& \lesssim \int_{2^l}^{2^{l+1}} \frac{1}{x^{1+\gamma}} |1 - e^{-(t-s) x^2 f(\varepsilon x)}| dx \lesssim \int_{2^l}^{2^{l+1}} \frac{1 \wedge |t-s| x^2}{x^{1+\gamma}} dx \\
& \lesssim (t-s)^{\gamma/2} \int_0^\infty \frac{1 \wedge z^2}{z^{1+\gamma}} dz \lesssim (t-s)^{\gamma/2}.
\end{aligned}$$

Let us now treat the case $\delta \leq \varepsilon 2^{l+1}$. Since $\delta m_{s,t}^{\gamma,\varepsilon}(x) = x^{-\gamma} \delta m_{s,t}^\varepsilon(x)$, we obtain

$$\begin{aligned}
\sum_{k=2^l}^{2^{l+1}-1} |\delta m_{s,t}^{\gamma,\varepsilon}(k) - \delta m_{s,t}^{\gamma,\varepsilon}(k+1)| & \leq \frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |\delta m_{s,t}^\varepsilon(k) - \delta m_{s,t}^\varepsilon(k+1)| \\
& + \frac{1}{2^{l\gamma}} \sup_{k \in [2^l, 2^{l+1}]} |\delta m_{s,t}^\varepsilon(k)|.
\end{aligned} \tag{6.21}$$

The second term in this expression can be bounded by

$$\begin{aligned}
\frac{1}{2^{l\gamma}} \sup_{k \in [2^l, 2^{l+1}]} |\delta m_{s,t}^\varepsilon(k)| & \lesssim \frac{1}{2^{l\gamma}} \sup_{k \in [2^l, 2^{l+1}]} \exp(-s k^2 \bar{f}(\varepsilon k)) |t-s|^{\gamma/2} k^\gamma \bar{f}(\varepsilon k)^{\gamma/2} \\
& \lesssim \sup_{k \in [2^l, 2^{l+1}]} \exp(-s k^2 \bar{f}(\varepsilon k)) |t-s|^{\gamma/2} \bar{f}(\varepsilon k)^{\gamma/2}.
\end{aligned}$$

As above, this expression can be estimated using the boundedness of $x \mapsto x^{\gamma/2} e^{-x}$ and the fact that $\varepsilon^2 < s$ by

$$\sup_{k \in [2^l, 2^{l+1}]} |t - s|^{\gamma/2} s^{-\gamma/2} k^{-\gamma} \lesssim \sup_{k \in [2^l, 2^{l+1}]} |t - s|^{\gamma/2} s^{-\gamma/2} \varepsilon^\gamma \lesssim |t - s|^{\gamma/2}.$$

It remains to bound the terms

$$\frac{1}{2^{l\gamma}} \sum_{k=2^l}^{2^{l+1}-1} |\delta m_{s,t}^\varepsilon(k) - \delta m_{s,t}^\varepsilon(k+1)| \leq \frac{1}{2^{l\gamma}} |\delta m_{s,t}^\varepsilon|_{\text{BV}}. \quad (6.22)$$

For this, it turns out to be sufficient to show that for $c \in (0, 1]$ we have the bounds

$$|G_c|_{L^\infty} \lesssim c, \quad |G_c|_{\text{BV}} \lesssim c, \quad (6.23)$$

where

$$G_c(x) \stackrel{\text{def}}{=} \exp(-x^2 f(x)) - \exp(-(1+c)x^2 f(x)).$$

Assuming that we have established (6.23), we can rewrite $\delta m_{s,t}^\varepsilon$ as

$$\delta m_{s,t}^\varepsilon(x) = \exp(-(s - \varepsilon^2)x^2 f(\varepsilon x)) G_{|t-s|/\varepsilon^2}(\varepsilon x).$$

By Assumption 1.2, both the BV and the supremum norms of the first factor are uniformly bounded so that, combining this with (6.23), we obtain the bound

$$|\delta m_{s,t}^\varepsilon(x)|_{\text{BV}} \lesssim \frac{|t-s|}{\varepsilon^2}.$$

On the other hand, Assumption 1.2 immediately implies that $|\delta m_{s,t}^\varepsilon(x)|_{\text{BV}} \lesssim 1$, so that $|\delta m_{s,t}^\varepsilon(x)|_{\text{BV}} \lesssim |t-s|^{\gamma/2} \varepsilon^{-\gamma}$. Plugging this back into (6.22) and using the fact that $\varepsilon 2^l \geq \delta/2$, we immediately obtain the required bound.

It remains to show (6.23). The bound on the supremum follows immediately from the inequality $e^{-y}(1 - e^{-cy}) \lesssim c$ by setting $y = x^2 f(x)$. To obtain the BV bound, set $A_c(x) = e^{-x} - e^{-(1+c)x}$ and note that one has the bound

$$\begin{aligned} |A_c(x) - A_c(y)| &= \left| \int_x^y e^{-z} - (1+c)e^{-(1+c)z} dz \right| \lesssim c \int_x^y e^{-\frac{z}{2}} dz \\ &= c |e^{-x/2} - e^{-y/2}|, \end{aligned}$$

so that the BV-norm of G_c is bounded by c times the BV-norm of $\exp(-\frac{1}{2}x^2 f(x))$. This on the other hand is bounded by Assumption 1.2, so that we have established (6.23). \square

APPENDIX A. ROUGH INTEGRALS

In this appendix we briefly summarise the definition and the properties of rough integrals we use. We refer the reader to [LQ02, Gub04, LCL07, HW10, FV10] for a more complete account of rough path theory.

As above for $X \in \mathcal{C}$ we will always use the notation $\delta X(x, y) := X(y) - X(x)$. For $R \in \mathcal{B}$ we will write $\delta R(x, y, z) := R(x, z) - R(x, y) - R(y, z)$. See [Gub04, GT10] for a discussion of the algebraic properties of these operators.

We want to define integrals of the type

$$\int_x^y Y(z) \otimes dZ(z) \quad (A.1)$$

for functions $Y, Z \in \mathcal{C}^\alpha$ for some $\alpha \in (0, 1)$. If $\alpha > \frac{1}{2}$ such integrals can be defined as limits of Riemann-sums of type

$$\sum_i Y(x_i) \otimes \delta Z(x_i, x_{i+1}). \quad (\text{A.2})$$

This yields the Young integral.

If $\alpha \leq \frac{1}{2}$ Riemann sums of type (A.2) will in general fail to converge. The idea is then to define a better approximation with the help of additional data. To this end we introduce the following definitions.

Definition A.1. An α -rough path consists of two functions $X \in \mathcal{C}^\alpha(\mathbb{R}^n)$ and $\mathbf{X} \in \mathcal{B}^{2\alpha}(\mathbb{R}^n \otimes \mathbb{R}^n)$ satisfying the relation

$$\mathbf{X}(x, z) - \mathbf{X}(x, y) - \mathbf{X}(y, z) = \delta X(x, y) \otimes \delta X(y, z) \quad (\text{A.3})$$

for all x, y, z . An α -rough path (X, \mathbf{X}) is called geometric if in addition for every x, y the symmetric part $\mathbf{X}^+(x, y) = \frac{1}{2}(\mathbf{X}(x, y) + \mathbf{X}(y, x))^T$ of $\mathbf{X}(x, y)$ satisfies

$$\mathbf{X}^+(x, y) = \frac{1}{2} \delta X(x, y) \otimes \delta X(x, y).$$

Following [Gub04] we also define:

Definition A.2. Let X be in \mathcal{C}^α A pair (Y, Y') with $Y \in \mathcal{C}^\alpha$ and $Y' \in \mathcal{C}^\alpha(\mathcal{L}(\mathbb{R}^n))$ is said to be controlled by X if for all x, y

$$\delta Y(x, y) = Y'(x) \delta X(x, y) + R_Y(x, y), \quad (\text{A.4})$$

with a remainder $R_Y \in \mathcal{B}^{2\alpha}$.

Note that (A.4) is a linear condition. So for a given X the space of paths that are controlled by X is a vector space.

Remark A.3. In general the decomposition (A.4) need not be unique, but in all of the situations we will encounter there is a natural choice of Y' .

If Y, Z are controlled by X and there is a choice of \mathbf{X} turning (X, \mathbf{X}) into a rough path, we construct the rough integral $\int Y dZ$ as the limit of the second order approximations

$$\sum_i Y(x_i) \otimes (Z(x_{i+1}) - Z(x_i)) + Y'(x_i) \mathbf{X}(x_i, x_{i+1}) Z'(x_i)^T. \quad (\text{A.5})$$

If $\alpha > \frac{1}{3}$, it turns out that these approximations converge:

Lemma A.4 ([Gub04, Thm 1 and Cor. 2]). Let $\alpha > \frac{1}{3}$. Suppose (X, \mathbf{X}) is an α rough path and Y, Z are controlled by X . Then the Riemann-sums defined in (A.5) converge as the mesh of the partition goes to zero. We call the limit rough integral and denote it by $\int Y(x) \otimes dZ(x)$.

The mapping $(Y, Z) \mapsto \int Y \otimes dZ$ is bilinear and we have the following bound:

$$\int_x^y Y(z) \otimes dZ(z) = Y(x) \otimes \delta Z(x, y) + Y'(x) \mathbf{X}(x, y) Z'(x)^T + Q(x, y), \quad (\text{A.6})$$

where the remainder satisfies

$$\begin{aligned} |Q|_{3\alpha} &\lesssim |R_Y|_{2\alpha} |Z|_\alpha + |Y'|_0 |X|_\alpha |R_Z|_{2\alpha} \\ &\quad + |\mathbf{X}|_{2\alpha} \left(|Y'|_\alpha |Z'|_C + |Y'|_C |Z'|_\alpha \right) + |X|_\alpha^2 |Y'|_0 |Z|_\alpha. \end{aligned} \quad (\text{A.7})$$

The rough integral also possesses continuity properties with respect to different rough paths. We refer the reader to [Gub04] for more details. The reason for using the notation $\overset{\circ}{\int}$ instead of \int is to keep a reminder of the fact that this is really an abuse of notation since $\overset{\circ}{\int} Y dZ$ also depends on the choices of Y' , Z' , and \mathbf{X} .

We will need the following scaling property of Young and rough integrals. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, recall the definition of the norm

$$|f|_{1,1} = \sum_{k \in \mathbb{Z}} \sup_{x \in [k, k+1]} (|f(x)| + |f'(x)|) \quad (\text{A.8})$$

given in Lemma 5.2. One then has the following bound:

Lemma A.5 (Lemma 5.1 in [HW10]). *Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function with finite $|f|_{1,1}$. Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$, (X, \mathbf{X}) be an α rough path and Y, Z be controlled by X . Then for any $\lambda > \lambda_0$*

$$\begin{aligned} \left| \overset{\circ}{\int}_{-\pi}^{\pi} f(\lambda x) Y(x) dZ(x) \right| &\lesssim |f|_{1,1} \lambda^{-\alpha} \left[|Y|_0 |Z|_{\alpha} + |R_Y|_{2\alpha} |Z|_{\alpha} \right. \\ &\quad \left. + |X|_{\alpha} |Y'|_{\mathcal{C}} |R_Z|_{2\alpha} + |\mathbf{X}|_{2\alpha} \left(|Y'|_0 |Z'|_{\mathcal{C}^{\alpha}} + |Y'|_{\mathcal{C}^{\alpha}} |Z'|_0 \right) + |X|_{\alpha}^2 |Y'|_{\mathcal{C}} |Z|_{\alpha} \right]. \end{aligned} \quad (\text{A.9})$$

Similar bounds also hold for differences of integrals against different rough paths (see [HW10, Rem. 5.3]).

APPENDIX B. REGULARITY RESULTS

We first quote a version of a classical regularity statement due to Garsia, Rodemich and Rumsey [GRR71]. For $R \in \mathcal{B}$ we will use the notation

$$\begin{aligned} |\delta R|_{[x,y]} &:= \sup_{x \leq z_1 < z_2 < z_3 \leq y} |\delta R(z_1, z_2, z_3)| \\ &= \sup_{x \leq z_1 < z_2 < z_3 \leq y} |R(z_1, z_3) - R(z_1, z_2) - R(z_2, z_3)|. \end{aligned}$$

Lemma B.1 (Lemma 4 in [Gub04]). *Suppose $R \in \mathcal{B}$. Denote by*

$$U = \int_{[-\pi, \pi]^2} \Theta \left(\frac{R(x, y)}{p(|x - y|/4)} \right) dx dy, \quad (\text{B.1})$$

where $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing function with $p(0) = 0$ and Θ is increasing, convex, and $\Theta(0) = 0$. Assume that there is a constant C such that

$$|\delta R|_{[x,y]} \leq \Theta^{-1} \left(\frac{C}{|y - x|^2} \right) p(|y - x|/4), \quad (\text{B.2})$$

for all x, y . Then

$$|R(x, y)| \lesssim \int_0^{|y-x|} \left[\Theta^{-1} \left(\frac{U}{r^2} \right) + \Theta^{-1} \left(\frac{C}{r^2} \right) \right] dp(r). \quad (\text{B.3})$$

for all x, y .

Remark B.2. We will use Lemma B.1 with $\Theta(u) = u^p$ and $p(x) = x^{\alpha+2/p}$. In this situation, if $R(x, y) = f(x) - f(y)$, Lemma B.1 states that

$$|f|_{\alpha} \lesssim \left(\int_{[-\pi, \pi]^2} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 2}} dx dy \right)^{1/p}, \quad (\text{B.4})$$

which is a version of Sobolev embedding theorem.

We will now proceed to extend this statement to derive bounds on functions that depend on several variables. We will use the abbreviated notation

$$\|F\|_{\mathcal{C}_T^{\alpha_1}(\mathcal{C}_2^{\alpha_2})} := \|F\|_{\mathcal{C}^{\alpha_1}([0,T], \mathcal{C}^{\alpha_2}[-\pi,\pi])},$$

and similarly for $\|R\|_{\mathcal{C}_T^{\alpha_1}(\mathcal{B}_2^{\alpha_2})}$.

Lemma B.3. *Let $\alpha_1, \alpha_2 > 0$, let $\gamma_1, \gamma_2 \in [0, 1]$ and let $p \geq 1$ be such that*

$$\alpha_1 < \gamma_1 \lambda_1 - \frac{1}{p}, \quad \alpha_2 < \gamma_2 \lambda_2 - \frac{1}{p} \quad (\text{B.5})$$

for some $\lambda_1, \lambda_2 > 0, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

(1) *Let F be a random function in $C([0, T], C[-\pi, \pi])$ satisfying*

$$\sup_{x \in [-\pi, \pi]} \mathbb{E} |F(t, x) - F(s, x)|^p \leq U_1^p |t - s|^{\gamma_1 p}, \quad (\text{B.6a})$$

$$\sup_{t \in [0, T]} \mathbb{E} |F(t, x) - F(t, y)|^p \leq U_2^p |x - y|^{\gamma_2 p}, \quad (\text{B.6b})$$

$$\sup_{\substack{x \in [-\pi, \pi] \\ t \in [0, T]}} \mathbb{E} |F(t, x)|^p \leq U_3^p. \quad (\text{B.6c})$$

Then we have

$$\mathbb{E} \|F\|_{\mathcal{C}_T^{\alpha_1}(\mathcal{C}^{\alpha_2})}^p \lesssim \left((U_1^{\lambda_1} + U_3^{\lambda_1})(U_2^{\lambda_2} + U_3^{\lambda_2})U_3^{\lambda_3} \right)^p. \quad (\text{B.7})$$

(2) *Similarly, let R be a random function in $C([0, T], \mathcal{B}[-\pi, \pi])$ satisfying*

$$\sup_{x, y \in [-\pi, \pi]} \mathbb{E} |R(t, x, y) - R(s, x, y)|^p \leq U_1^p |t - s|^{\gamma_1 p}, \quad (\text{B.8a})$$

$$\sup_{t \in [0, T]} \mathbb{E} |R(t, x, y)|^p \leq U_2^p |x - y|^{\gamma_2 p}, \quad (\text{B.8b})$$

$$\sup_{t \in [0, T]} \mathbb{E} |\delta R(t)|_{[x, y]}^p \leq U_2^p |x - y|^{\gamma_2 p}, \quad (\text{B.8c})$$

$$\sup_{\substack{x, y \in [-\pi, \pi] \\ t \in [0, T]}} \mathbb{E} |R(t, x, y)|^p \leq U_3^p. \quad (\text{B.8d})$$

Then we have

$$\mathbb{E} \|R\|_{\mathcal{C}_T^{\alpha_1}(\mathcal{B}^{\alpha_2})}^p \lesssim \left((U_1^{\lambda_1} + U_3^{\lambda_1})(U_2^{\lambda_2} + U_3^{\lambda_2})U_3^{\lambda_3} \right)^p. \quad (\text{B.9})$$

Proof. Let us start by proving (B.7). To this end for fixed $0 \leq s < t \leq T$ we introduce the notation

$$F_{s,t}(x) = F(t, x) - F(s, x).$$

We have to bound the quantity

$$\mathbb{E} \|F\|_{\mathcal{C}_T^{\alpha_1}(\mathcal{C}^{\alpha_2})}^p \lesssim \mathbb{E} \left(\sup_{0 \leq s < t \leq T} \frac{|F_{s,t}|_{\mathcal{C}^{\alpha_2}}}{|t - s|^{\alpha_1}} \right)^p + \mathbb{E} |F(0, \cdot)|_{\mathcal{C}^{\alpha_2}}^p. \quad (\text{B.10})$$

To this end for fixed s, t we can write

$$\mathbb{E} |F_{s,t}|_{\mathcal{C}^{\alpha_2}}^p \lesssim \mathbb{E} \left(\sup_{x \neq y \in [-\pi, \pi]} \frac{|F_{s,t}(y) - F_{s,t}(x)|}{|x - y|^{\alpha_2}} \right)^p + \mathbb{E} |F_{s,t}(0)|^p. \quad (\text{B.11})$$

For the first term we get using the Garsia-Rodemich-Rumsey Lemma B.1 (see also Remark B.2)

$$\begin{aligned} & \mathbb{E} \left(\sup_{x,y \in [-\pi, \pi]} \frac{|F_{s,t}(y) - F_{s,t}(x)|}{|x - y|^{\alpha_2}} \right)^p \\ & \lesssim \mathbb{E} \left[\int_{[-\pi, \pi]^2} \frac{|F_{s,t}(x) - F_{s,t}(y)|^p}{|x - y|^{\alpha_2 p + 2}} dx dy \right] \\ & = \int_{[-\pi, \pi]^2} \frac{1}{|x - y|^{\alpha_2 p + 2}} \mathbb{E} \left(F_{s,t}(x) - F_{s,t}(y) \right)^p dx dy. \end{aligned} \quad (\text{B.12})$$

Using Hölder inequality the expectation in the last integral can be bounded by

$$\begin{aligned} \mathbb{E} |F_{s,t}(x) - F_{s,t}(y)|^p & \lesssim \sup_{z \in \{x, y\}} (\mathbb{E} |F(t, z) - F(s, z)|^p)^{\lambda_1} \\ & \quad \times \sup_{r \in \{s, t\}} (\mathbb{E} |F(r, x) - F(r, y)|^p)^{\lambda_2} \\ & \quad \times \sup_{\substack{z \in \{x, y\} \\ r \in \{s, t\}}} (\mathbb{E} |F(r, z)|^p)^{\lambda_3} \\ & \leq (U_1^p |t - s|^{p\gamma_1})^{\lambda_1} (U_2^p |x - y|^{p\gamma_2})^{\lambda_2} (U_3^p)^{\lambda_3}. \end{aligned} \quad (\text{B.13})$$

Here in the last step we have made use of the bounds (B.6a), (B.6b), and (B.6c). Similarly, according to the assumption (B.6a) and (B.6c) we get for the second term on the right-hand side of (B.11)

$$\mathbb{E} |F_{s,t}(0)|^p \lesssim (U_1^p |t - s|^{p\gamma_1})^{\lambda_1} (U_3^p)^{\lambda_2 + \lambda_3}. \quad (\text{B.14})$$

Therefore, we get

$$\begin{aligned} \mathbb{E} |F_{s,t}|_{\mathcal{C}^{\alpha_2}}^p & \lesssim (U_1^{\lambda_1} U_3^{\lambda_3} |t - s|^{\gamma_1 \lambda_1})^p \\ & \quad \times \left(U_3^{p\lambda_2} + U_2^{p\lambda_2} \int_{[-\pi, \pi]^2} \frac{|x - y|^{p\gamma_2 \lambda_2}}{|x - y|^{\alpha_2 p + 2}} dx dy \right). \end{aligned} \quad (\text{B.15})$$

The integral appearing in (B.15) is finite if and only if α_2 satisfies the condition given in (B.5). So in that case we get

$$\mathbb{E} |F_{s,t}|_{\mathcal{C}^{\alpha_2}}^p \lesssim (U_1^{\lambda_1} U_3^{\lambda_3} |t - s|^{\gamma_1 \lambda_1})^p (U_2^{\lambda_2} + U_3^{\lambda_2})^p. \quad (\text{B.16})$$

Then, to get uniform bounds in s, t we apply the Garsia-Rodemich-Rumsey Lemma to the first term on the right-hand side of (B.10)

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s, t \leq T} \frac{|F_{s,t}|_{\mathcal{C}^{\alpha_2}}}{|t - s|^{\alpha_1}} \right)^p & \lesssim \int_{[0, T]^2} \frac{\mathbb{E} |F_{s,t}|_{\mathcal{C}^{\alpha_2}}^p}{|t - s|^{\alpha_1 p + 2}} ds dt \\ & \lesssim (U_1^{\lambda_1} U_3^{\lambda_3})^p (U_2^{\lambda_2} + U_3^{\lambda_2})^p \int_{[0, T]^2} \frac{|t - s|^{\gamma_1 \lambda_1 p}}{|t - s|^{\alpha_1 p + 2}} ds dt. \end{aligned} \quad (\text{B.17})$$

The integral appearing on the right-hand side of (B.17) is finite if and only if α_1 satisfies the condition (B.5). Then we get

$$\mathbb{E} \left(\sup_{0 \leq s, t \leq T} \frac{|F_{s,t}|_{\mathcal{C}^{\alpha_2}}}{|t - s|^{\alpha_1}} \right)^p \lesssim (U_1^{\lambda_1} U_3^{\lambda_3})^p (U_2^{\lambda_2} + U_3^{\lambda_2})^p.$$

Finally, to conclude it only remains to bound the term $\mathbb{E}|F(0, \cdot)|_{C^{\alpha_2}}^p$ on the right-hand side of (B.10). This can be done by observing that

$$\begin{aligned} \mathbb{E}|F(0, \cdot)|_{C^{\alpha_2}}^p &\lesssim \mathbb{E}|F(0, 0)|^p \\ &\quad + \int_{[-\pi, \pi]^2} \frac{1}{|x - y|^{\alpha_2 p + 2}} \mathbb{E}|F(0, x) - F(0, y)|^p dx dy \\ &\lesssim U_3^p + U_3^{(\lambda_1 + \lambda_3)p} U_2^{\lambda_2 p} \\ &\lesssim U_3^{(\lambda_1 + \lambda_3)p} (U_2^{\lambda_2} + U_3^{\lambda_2})^p. \end{aligned}$$

This finishes the proof of (B.7).

The proof of (B.9) is very similar and we only sketch it. As above we will use the notation

$$R_{s,t}(x, y) = R(t; x, y) - R(s; x, y).$$

Similarly to (B.10) we need to derive a bound on

$$\mathbb{E}\|R\|_{C_T^{\alpha_1}(\mathcal{B}^{\alpha_2})}^p \lesssim \mathbb{E}\left(\sup_{0 \leq s < t \leq T} \frac{1}{|t - s|^{\alpha_1}} |R_{s,t}|_{\alpha_2}\right)^p + \mathbb{E}|R(0; \cdot, \cdot)|_{\alpha_2}^p. \quad (\text{B.18})$$

For fixed s, t we get using Gubinelli's version of the Garsia-Rodemich-Rumsey inequality B.1

$$\mathbb{E}|R_{s,t}|_{\alpha_2}^p \lesssim \int_{[-\pi, \pi]^2} \frac{\mathbb{E}[|R_{s,t}(x, y)|^p + |\delta R_{s,t}|_{[x,y]}^p]}{|x - y|^{\alpha_2 p + 2}} dx dy. \quad (\text{B.19})$$

The difference with respect to the case of F is the appearance of the extra term $|\delta R|_{[x,y]}$. On the other side there is no lower order term in the \mathcal{B}^{α_2} norm. Then using Hölder inequality and the bounds (B.8a), (B.8b), (B.8c) and (B.8d), and then the integrability condition (B.5) in the same way as in (B.12) and (B.13), the expectation in the right-hand side of (B.19) can be bounded by

$$(U_1^p |t - s|^{p\gamma_1})^{\lambda_1} (U_2^p |x - y|^{p\gamma_2})^{\lambda_2} (U_3^p)^{\lambda_3}.$$

Plugging this back in (B.18) we get as in (B.17),

$$\begin{aligned} &\mathbb{E}\left(\sup_{0 \leq s < t \leq T} \frac{|R_{s,t}|_{\alpha_2}}{|t - s|^{\alpha_1}}\right)^p \\ &\lesssim (U_1^{\lambda_1} U_3^{\lambda_3})^p \cdot (U_2^{\lambda_2} + U_3^{\lambda_2})^p \int_{[0, T]^2} \frac{|t - s|^{\gamma_1 \lambda_1 p}}{|t - s|^{\alpha_1 p + 2}} ds dt \\ &\lesssim (U_1^{\lambda_1} U_3^{\lambda_3})^p \cdot (U_2^{\lambda_2} + U_3^{\lambda_2})^p. \end{aligned}$$

Then applying [Gub04, Lem. 4] once more we can bound the second term appearing on the right-hand side of (B.18) by

$$\mathbb{E}|R(0; \cdot, \cdot)|_{\alpha_2}^p \lesssim (U_2^{\lambda_2} U_3^{\lambda_1 + \lambda_3})^p.$$

This finishes the proof of (B.9). \square

In a similar spirit is the following Banach space-valued version of Kolmogorov's continuity criterion, which is slightly more convenient in some cases.

Lemma B.4. *Let $(\varphi(t))_{t \in [0, T]}$ be a Banach space-valued random field having the property that for any $q \in (2, \infty)$ there exists a constant $K_q > 0$ such that*

$$\begin{aligned} (\mathbb{E} \|\varphi(t)\|^q)^{\frac{1}{q}} &\leq K_q (\mathbb{E} \|\varphi(t)\|^2)^{\frac{1}{2}}, \\ (\mathbb{E} \|\varphi(s) - \varphi(t)\|^q)^{\frac{1}{q}} &\leq K_q (\mathbb{E} \|\varphi(s) - \varphi(t)\|^2)^{\frac{1}{2}}, \end{aligned} \tag{B.20}$$

for all $s, t \in [0, T]$. Furthermore, suppose that the estimate

$$\mathbb{E} \|\varphi(s) - \varphi(t)\|^2 \leq K_0 |s - t|^\delta$$

holds for some $K_0, \delta > 0$ and all $s, t \in [0, T]$. Then, for every $p > 0$ there exists $C > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|\varphi(t)\|^p \leq C (K_0 + \mathbb{E} \|\varphi(0)\|^2)^{\frac{p}{2}}.$$

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